MEASURE THEORY

1. Preliminaries

For any given sequence (x_n) in the two-point compactification $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ of the reals $\mathbb{R}$

$$\liminf x_n = \sup_m \inf_{n \geq m} x_n,$$
$$\limsup x_n = \inf_m \sup_{n \geq m} x_n.$$

We also have a natural order $<$ on $\overline{\mathbb{R}}$, by extending the one on $\mathbb{R}$ and by letting $-\infty < x < +\infty$ for all $x \in \mathbb{R}$.

We make note that $\liminf x_n \leq \limsup x_n$ and that $\liminf (-x_n) = -\limsup x_n$.

Theorem 1.1. *Let (x_n) be a sequence in $\overline{\mathbb{R}}$. Then there exists subsequences $\left(x_{n_j}\right)$ and (x_{n_k}) with*

$$x_{n_j} \to \liminf x_n$$
$$x_{n_k} \to \limsup x_n.$$

Further, if (x_{n_ℓ}) is an arbitrary convergent subsequence then

$$\liminf x_n \leq \lim x_{n_\ell} \leq \limsup x_n.$$

The $\liminf$ *is the smallest value with a subsequence converging to it.*

Proof. Without loss of generality, consider $M = \limsup x_n$. Consider first $M = -\infty$, so we have $M_n = \sup_{k \geq n} x_k \to -\infty$. Choose n_1 for which $x_{m_1} \leq M_{n_1} < -1$. Choose $n_2 > n_1$ such that $x_{m_2} \leq M_{n_2} < -2$ and so on.

If on the other hand $-\infty < M < \infty$, one can choose $M_k - \frac{1}{k} \leq x_{n_k} \leq M_k$.

Lastly, if $M = \infty$, then we have $M_n = \sup_{k \geq n} x_k \to \infty$. We choose $k \leq x_{n_k}$ for each k, which will clearly be possible (otherwise x_n is bounded for sufficiently large n and it would follow $M \neq \infty$). □

2. Measurable Functions

Definition 2.1. A collection $\mathscr{A}$ of subsets of a set X is said to be a σ-*algebra*, or a σ-*field*, if:

1. $\mathscr{A}$ is closed under complementation.
2. $\mathscr{A}$ is closed under countable unions.

An ordered pair $\langle X, \mathscr{A} \rangle$ consisting of a set X and a σ-algebra $\mathscr{A}$ of X is called a *measurable space*. Members of $\mathscr{A}$ are said to be *measurable*.

Note. An immediate consequence is that given any σ-algebra $\mathscr{A}$ of a set X, is that both $\varnothing$ and X are measurable. For $\varnothing$ is a countable collection, giving us $\bigcup \varnothing = \varnothing$ is a member of $\mathscr{A}$ (and consequently so is X, being the complement of $\varnothing$). One can further deduce by De Morgan's Laws that a σ-algebra is closed under countable intersections.

Let $\mathscr{A}$ be a non-empty collection of subsets of X. We observe there is a smallest σ-algebra containing $\mathscr{A}$. To see this, certainly the collection containing all subsets of X is a σ-algebra of X which contains $\mathscr{A}$. The intersection of all σ-algebras of X

containing $\mathscr{A}$ is the desired σ-algebra of X. This is called the *σ-algebra generated by $\mathscr{A}$*.

Let X be the set $\mathbb{R}$ of real numbers. The *Borel algebra* $\mathscr{B}$ is the σ-algebra generated by all open intervals (a, b) in X. Observe that the Borel algebra $\mathscr{B}$ is also the σ-algebra generated by all closed intervals $[a, b]$ in X. A member of $\mathscr{B}$ is called a *Borel set*.

Definition 2.2. A function f on X to $\mathbb{R}$ is said to be *$\mathscr{A}$-measurable*, or simply *measurable*, if for every real number α the set

$$\{x \in X \mid f(x) > \alpha\}$$

belongs to $\mathscr{A}$.

Lemma 2.3. *The following statements are equivalent for a function f on X to $\mathbb{R}$:*

1. *For every $\alpha \in \mathbb{R}$, the set $A_\alpha = \{x \in X \mid f(x) > \alpha\}$ belongs to $\mathscr{A}$.*
2. *For every $\alpha \in \mathbb{R}$, the set $B_\alpha = \{x \in X \mid f(x) \le \alpha\}$ belongs to $\mathscr{A}$.*
3. *For every $\alpha \in \mathbb{R}$, the set $C_\alpha = \{x \in X \mid f(x) \ge \alpha\}$ belongs to $\mathscr{A}$.*
4. *For every $\alpha \in \mathbb{R}$, the set $D_\alpha = \{x \in X \mid f(x) < \alpha\}$ belongs to $\mathscr{A}$.*

Proof. As $\mathscr{A}$ is closed under complementation, it is clear that (1) and (2) are equivalent and that (3) and (4) are equivalent.

Suppose (1) holds. Fixing α, we see that $A_{\alpha - \frac{1}{n}}$ is measurable for each positive integer n. Consequently, $C_\alpha = \bigcap_{n=1}^\infty A_{\alpha-\frac{1}{n}}$ is measurable, giving us (1) implies (3).

Conversely, suppose (3) holds. Fixing α, we see that $C_{\alpha+\frac{1}{n}}$ is measurable for each positive integer n. Hence, $A_\alpha = \bigcup_{n=1}^\infty C_{\alpha+\frac{1}{n}}$ is measurable, and thus we are done. □

Proposition 2.4. *If $f, g : X \to \mathbb{R}$ are $\mathscr{S}$-measurable, then so are the following:*

1. *fg;*
2. *af for $a \in \mathbb{R}$;*
3. *$|f|^\alpha$ for $\alpha \in \mathbb{R}^+$;*
4. *$f + g$;*
5. *if f_n is a sequence of $\mathscr{S}$-measurable functions, then* $\limsup f_n$ *and* $\liminf f_n$ *are also.*

Proof. For fg, we consider

$$fg = \frac{1}{4}\Big((f+g)^2 - (f-g)^2\Big).$$

For af, simply split into the three cases.

For $f + g$, take

$$\{x \in X \mid f(x) + g(x) > a\} = \bigcup_{r \in \mathbb{Q}} \{x \in X \mid f(x) > r\} \cap \{x \in X \mid g(x) > a - r\}.$$

For (5), for supremum take union, and infimum take intersection. □

Definition 2.5. An extended real-valued function f on X is *$\mathscr{A}$-measurable* in case the set $\{x \in X \mid f(x) > \alpha\}$ belongs to $\mathscr{A}$ for each real number α. The collection of all extended real-valued $\mathscr{A}$-measurable functions on X is denoted by $M(X, \mathscr{A})$.

Lemma 2.6. *An extended real-valued function f is measurable if and only if the sets*

$$A = \{x \in X \mid f(x) = +\infty\}$$
$$B = \{x \in X \mid f(x) = -\infty\}$$

belong to $\mathscr{A}$ and the real-valued function f_1 defined by

$$f_1(x) = f(x), \text{ if } x \notin A \cup B,$$
$$= 0, \text{ if } x \in A \cup B,$$

is measurable.

Note. Observe that if $f \in M(X, \mathscr{A})$, then

$$\{x \in X \mid f(x) = +\infty\} = \bigcap_{n\in\mathbb{N}} \{x \in X \mid f(x) > n\}, \text{ and}$$
$$\{x \in X \mid f(x) = -\infty\} = \mathscr{C}\left(\bigcup_{n\in\mathbb{N}} \{x \in X \mid f(x) > -n\}\right).$$

Proof. ($\Longrightarrow$) Suppose f is measurable. It is immediate that both A and B belong to $\mathscr{A}$ by definition. Now, fix $\alpha \in \mathbb{R}$. Let

- $A_\alpha = \{x \in X \mid f_1(x) > \alpha\}$,
- $C_\alpha = \{x \in X \mid f(x) > \alpha\}$.

Observe that

$$A_\alpha = \begin{cases} C_\alpha \cup B & \text{if } \alpha < 0; \\ C_\alpha \setminus A & \text{if } \alpha \geq 0; \end{cases}$$

giving us A_α belongs to $\mathscr{A}$. It follows f_1 is measurable.

($\Longleftarrow$) Suppose $\alpha \in \mathbb{R}$. Observe that

$$C_\alpha = \begin{cases} A_\alpha \setminus B & \text{if } \alpha < 0; \\ A_\alpha \cup A & \text{if } \alpha \geq 0; \end{cases}$$

giving us C_α belongs to $\mathscr{A}$. It follows f is measurable. □

Lemma 2.7. *Let (f_n) be a sequence in $M(X, \mathscr{A})$ and define the functions*

$$f(x) = \inf f_n(x), \ F(x) = \sup f_n(x),$$
$$f^*(x) = \liminf f_n(x), \ F^*(x) = \limsup f_n(x).$$

Then f, F, f^* and F^* belong to $M(X, \mathscr{A})$.

Proof. Observe that

$$\{x \in X \mid f(x) \geq \alpha\} = \bigcap_{n\in\mathbb{N}} \{x \in X \mid f_n(x) \geq \alpha\}, \text{ and}$$
$$\{x \in X \mid F(x) > \alpha\} = \bigcup_{n\in\mathbb{N}} \{x \in X \mid f_n(x) > \alpha\},$$

which gives us that f and F are both measurable. From this, by definition, it easily follows that f^* and F^* are measurable. □

Corollary 2.8. *If (f_n) is a sequence in $M(X, \mathscr{A})$ which converges to f on X, then f is in $M(X, \mathscr{A})$.*

Proof. In this case $f(x) = \lim f_n(x) = \liminf f_n(x)$. □

Lemma 2.9. *If f is a non-negative function in $M(X, \mathscr{A})$, then there exists a sequence (φ_n) in $M(X, \mathscr{A})$ such that*

1. *$0 \leq \varphi_n(x) \leq \varphi_{n+1}(x)$ for $x \in X$, $n \in \mathbb{N}$.*
2. *$f(x) = \lim \varphi_n(x)$ for each $x \in X$.*
3. *Each φ_n has only a finite number of real values.*

Proof. Let n be a fixed natural number. If $k = 0, 1, \dots, n2^n - 1$, let

$$E_{kn} = \big\{x \in X \;\big|\; k2^{-n} \leq f(x) < (k+1)2^{-n}\big\},$$

and if $k = n2^n$, let E_{kn} be the set $\{x \in X \mid f(x) \geq n\}$. We observe that the sets E_{kn} are disjoint, belong to $\mathscr{A}$, and have union equal to X. If we define φ_n to be equal to $k2^{-n}$ on E_{kn}, then φ_n belongs to $M(X, \mathscr{A})$. It is readily established that properties (1), (2) and (3) hold. □

Proposition 2.10. *Let $\mathscr{A}$ be a σ-algebra on $\mathbb{R}$. Then $\mathscr{A}$ contains all open intervals if and only if it contains all closed intervals.*

Proof. ($\Longrightarrow$) Observe that $[a,b] = \bigcap_{n\in\mathbb{N}} \left(a - \frac{1}{n}, b + \frac{1}{n}\right)$, since $a \leq x \leq b$ holds iff $a - \frac{1}{n} < x < b + \frac{1}{n}$ holds for all $n \in \mathbb{N}$. This establishes sufficiency by using definition of σ-algebra along with De Morgan's Laws.

($\Longleftarrow$) Similarly, one observes that $(a,b) = \bigcup_{n\in\mathbb{N}} [a + \frac{1}{n}, b - \frac{1}{n}]$, since $a < x < b$ holds iff $a + \frac{1}{n} \leq x \leq b - \frac{1}{n}$ holds for some $n \in \mathbb{N}$. This establishes necessity by using definition of σ-algebra. □

Proposition 2.11. *The Borel algebra $\mathscr{B}$ can be generated by the collection of all half-open intervals $(a, b]$. It can also be generated by the collection of all half-rays $(a, \infty) = \{x \in \mathbb{R} \mid x > a\}$.*

Proof. Firstly, observe that $(a,b] = \bigcap_{n\in\mathbb{N}} \left(a, b + \frac{1}{n}\right)$, since $a < x \leq b$ holds iff $a < x < b + \frac{1}{n}$ holds for all $n \in \mathbb{N}$. It follows that $(a,b]$ is a member of $\mathscr{B}$. Similarly, $(a,b) = \bigcup_{n\in\mathbb{N}} (a, b - \frac{1}{n}]$, since $a < x < b$ holds iff $a < x \leq b - \frac{1}{n}$ for some $n \in \mathbb{N}$. This completes the first part of the proof.

Next, notice that $(a, \infty) = \bigcup_{n\in\mathbb{N}} (a, n)$, giving us that (a, ∞) is a member of $\mathscr{B}$. We also have that $(a, b] = (a, \infty) \cap X \setminus (b, \infty)$, which means all half-open intervals are in the σ-algebra generated by all half-rays. That is, $\mathscr{B}$ is contained in the σ-algebra generated by all half-rays. □

Proposition 2.12. *Let (A_n) be a sequence of subsets of a set X. Let $E_0 = \varnothing$ and for each $n \in \mathbb{N}$, define*

$$E_n = \bigcup_{k=1}^{n} A_k$$
$$F_n = A_n \setminus E_{n-1}.$$

Then (E_n) is a monotone increasing sequence of sets and (F_n) is a disjoint sequence of sets such that

$$\bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} F_n = \bigcup_{n\in\mathbb{N}} A_n.$$

Proof. Suppose $n < m$. Then $n \leq m - 1$, implying that F_m is disjoint with A_n because $A_n \subseteq E_n \subseteq E_{m-1}$. Consequently, F_n and F_m are disjoint. It follows that (F_n) is indeed a disjoint sequence of sets.

It is clear $\bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} A_n$. Also, $F_n = \bigcap_{k=1}^{n} A_n \setminus A_k$, which obviously partitions $\bigcup_{n\in\mathbb{N}} A_n$. For if $x \in \bigcup_{n\in\mathbb{N}} A_n$, then there exists minimal k such that $x \in A_k$. We have that $x \in F_k$ for such k. □

Proposition 2.13. *Let (A_n) be a sequence of subsets of a set X. If A consists of all $x \in X$ which belong to infinitely many of the sets A_n, then*

$$A = \limsup A_n = \bigcap_{m=1}^{\infty} \left(\bigcup_{n=m}^{\infty} A_n \right).$$

Proof. Suppose $x \in A$. Then for each $m \in \mathbb{N}$, there exists $n \geq m$ such that $x \in A_n$. It follows that $x \in \bigcup_{n=m}^{\infty} A_n$. As m was arbitrary, $x \in \limsup A_n$.

Conversely, suppose $x \in \limsup A_n$. Then for any given $m \in \mathbb{N}$, $x \in \bigcup_{n=m}^{\infty} A_n$. That is to say, $x \in A_n$ for some $n \geq m$. Clearly, this gives us that x belongs to infinitely many of the sets A_n and so is contained in A. □

Proposition 2.14. *Let (A_n) be a sequence of subsets of a set X. If B consists of all $x \in X$ which belong to all but a finite number of the sets A_n, then*

$$B = \liminf A_n = \bigcup_{m=1}^{\infty} \left(\bigcap_{n=m}^{\infty} A_n \right).$$

Proof. Suppose $x \in B$. Then there exists $m \in \mathbb{N}$ such that $x \in A_n$ for all $n \geq m$. This clearly gives us $x \in \liminf A_n$. The converse is also obvious. □

Proposition 2.15. *If (E_n) is a sequence of subsets of a set X which is monotone increasing, then*

$$\limsup E_n = \bigcup_{n \in \mathbb{N}} E_n = \liminf E_n.$$

Proof. Observe that

$$\begin{aligned} \limsup E_n &= \bigcap_{m=1}^{\infty} \left(\bigcup_{n=m}^{\infty} E_n \right) \\ &= \bigcap_{m=1}^{\infty} \left(\bigcup_{n=1}^{\infty} E_n \right) \\ &= \bigcup_{n=1}^{\infty} E_n \\ &= \bigcup_{n=1}^{\infty} \left(\bigcap_{m=n}^{\infty} E_m \right) \\ &= \liminf E_n. \end{aligned}$$

□

Proposition 2.16. *If (F_n) is a sequence of subsets of a set X which is monotone decreasing, then*

$$\limsup F_n = \bigcap_{n \in \mathbb{N}} F_n = \liminf F_n.$$

Proof. Observe that

$$\begin{aligned} \limsup F_n &= \bigcap_{m=1}^{\infty} \left(\bigcup_{n=m}^{\infty} F_n \right) \\ &= \bigcap_{m=1}^{\infty} F_m \\ &= \bigcup_{n=1}^{\infty} \left(\bigcap_{m=n}^{\infty} F_m \right) \\ &= \liminf F_n. \end{aligned}$$

□

Proposition 2.17. *If (A_n) is a sequence of subsets of X,*

$$\varnothing \subseteq \liminf A_n \subseteq \limsup A_n \subseteq X.$$

Proof. Suppose $x \in \liminf A_n$. Then there exists $m \in \mathbb{N}$ such that $x \in A_n$ for all $n \geq m$. Fix the natural number m^*. It is clear that $x \in A_n$ for some $n \geq m^*$. From this, one gets that $x \in \limsup A_n$.

An example of a sequence (A_n) such that $\liminf A_n = \varnothing$ and $\limsup A_n = X$ is the following. Let $X = \{0, 1\}$ and define

$$A_n = \begin{cases} \{0\} & \text{if } n \text{ is even;} \\ \{1\} & \text{if } n \text{ is odd.} \end{cases}$$

One can easily obtain that $\liminf A_n = \varnothing$ and that $\limsup A_n = X$.

We now provide an example of a sequence (A_n) which is neither monotone increasing or decreasing, but is such that

$$\lim A_n = \liminf A_n = \limsup A_n.$$

Consider $X = \{0\} \cup \{p \mid p \text{ is prime}\}$ and define

$$A_n = \begin{cases} \{0, n\} & \text{if } n \text{ is prime;} \\ \{0\} & \text{otherwise.} \end{cases}$$

We have that $\liminf A_n = \limsup A_n = \{0\}$. □

Example 2.18. We provide an example of a function $f : X \to \mathbb{R}$ which is not $\mathscr{A}$-measurable, but is such that the functions $|f|$ and f^2 are $\mathscr{A}$-measurable. Consider $X = \{-1, 1\}$, $\mathscr{A} = \{\varnothing, X\}$, and $f(-1) = -1$, $f(1) = 1$. We have that f is not $\mathscr{A}$-measurable, since $\{1\}$ is not measurable. Yet, $|f|$ and f^2 are both $\mathscr{A}$-measurable.

Proposition 2.19. *Let* $\operatorname{mid}(a, b, c)$ *denote the value in the middle. Then*

$$\operatorname{mid}(a, b, c) = \inf(\sup(a, b), \sup(a, c), \sup(b, c)).$$

Proof. Without loss of generality, suppose $a \leq b \leq c$. Then

$$b = \operatorname{mid}(a, b, c) = \inf(\sup(a, b), \sup(a, c), \sup(b, c)) = \inf(b, c) = b.$$

□

Proposition 2.20. *If* f_1, f_2, f_3 *are* $\mathscr{A}$*-measurable functions on* X *to* $\mathbb{R}$ *and if* g *is defined for all* $x \in X$ *by*

$$g(x) = \operatorname{mid}(f_1(x), f_2(x), f_3(x)),$$

then g *is* $\mathscr{A}$*-measurable.*

Proof. Suppose $\alpha \in \mathbb{R}$, and let $A = \{x \in X \mid g(x) > \alpha\}$ and $B_i = \{x \in X \mid f_i(x) > \alpha\}$ for each $i = 1, 2, 3$. Fix $x \in A$. It follows that x is a member of at least two of B_1, B_2 and B_3. Hence, $A \subseteq \bigcap_{i \neq j} B_i \cup B_j$. Conversely, if $x \in \bigcap_{i \neq j} B_i \cup B_j$, one easily deduces that $x \in A$. □

Proposition 2.21. *If* f *is measurable and* $A > 0$*, then the truncation* f_A *defined by*

$$\begin{aligned} f_A(x) &= f(x), \quad \textit{if } |f(x)| \leq A, \\ &= A, \quad \textit{if } f(x) > A, \\ &= -A, \;\; f(x) < -A, \end{aligned}$$

is measurable.

Proof. Suppose $\alpha \in \mathbb{R}$, and let

- $Y = \{x \in X \mid f_A(x) > \alpha\}$,
- $B = \{x \in X \mid f(x) > \alpha\}$.

Then

$$Y = \begin{cases} X & \text{if } \alpha < -A; \\ B & \text{if } -A \le \alpha < A; \\ \varnothing & \text{if } \alpha \ge A; \end{cases}$$

gives us that Y is measurable. □

Proposition 2.22. *Let f be a function defined on a set X with values in a set Y. Let $\mathscr{X}$ be a σ-algebra of subsets of X and let $\mathscr{Y} = \{E \subseteq Y \mid f^{-1}(E) \in \mathscr{X}\}$. Then $\mathscr{Y}$ is a σ-algebra.*

Proof. Suppose $E \in \mathscr{Y}$. Then $f^{-1}(E) \in \mathscr{X}$, implying that $X \backslash f^{-1}(E) = f^{-1}(Y \setminus E)$ is measurable. It follows that $Y \setminus E \in \mathscr{Y}$, therefore giving us that it is closed under complementation. Now, suppose that $\mathscr{A}$ is a countable subcollection of $\mathscr{Y}$. Then $f^{-1}(\mathscr{A})$ is a countable subcollection of $\mathscr{X}$. Hence, $\bigcup f^{-1}(\mathscr{A}) = f^{-1}(\bigcup \mathscr{A})$ is a member of $\mathscr{X}$. Thus, $\bigcup \mathscr{A} \in \mathscr{Y}$, which gives us $\mathscr{Y}$ is a σ-algebra. □

Proposition 2.23. *Let $(X, \mathscr{X})$ be a measurable space and f be defined on X to Y. Let $\mathscr{A}$ be a collection of subsets of Y such that $f^{-1}(E) \in \mathscr{X}$ for every $E \in \mathscr{A}$. Then $f^{-1}(F) \in \mathscr{X}$ for any set F which belongs to the σ-algebra generated by $\mathscr{A}$.*

Proof. The σ-algebra generated by $\mathscr{A}$ can be viewed as the intersection over all σ-algebras containing $\mathscr{A}$. As $\{E \subseteq Y \mid f^{-1}(E) \in \mathscr{X}\}$ is a σ-algebra containing $\mathscr{A}$, the result readily follows. □

Proposition 2.24. *Let $(X, \mathscr{X})$ be a measurable space and f be a real-valued function defined on X. Then f is $\mathscr{X}$-measurable if and only if $f^{-1}(E) \in \mathscr{X}$ for every Borel set E.*

Proof. One needs only use the half-rays definition for generating the Borel σ-algebra. □

Proposition 2.25. *Let $(X, \mathscr{X})$ be a measurable space, f be an $\mathscr{X}$-measurable function on X to $\mathbb{R}$ and let φ be a continuous function on $\mathbb{R}$ to $\mathbb{R}$. Then $\varphi \circ f$, defined by $(\varphi \circ f)(x) = \varphi(f(x))$, is $\mathscr{X}$-measurable.*

Proof. Suppose E is a Borel set. Then $\varphi^{-1}(E)$ is a Borel set, since E can be expressed as the arbitrary union of open and closed sets in $\mathbb{R}$. Then $f^{-1}(\varphi^{-1}(E))$ must be a member of $\mathscr{X}$, since f is $\mathscr{X}$-measurable. Observe that $f^{-1}(\varphi^{-1}(E)) = (\varphi \circ f)^{-1}(E)$, giving us $\varphi \circ f$ is $\mathscr{X}$-measurable, as desired. □

A non-empty collection $\mathscr{M}$ of subsets of a set X is called a *monotone class* if, for each monotone increasing sequence (E_n) in $\mathscr{M}$ and each monotone decreasing sequence (F_n) in $\mathscr{M}$, the sets

$$\bigcup_{n \in \mathbb{N}} E_n, \quad \bigcap_{n \in \mathbb{N}} F_n$$

belong to $\mathscr{M}$.

Proposition 2.26. *A σ-algebra is a monotone class. Also, if $\mathscr{A}$ is a non-empty collection of subsets of X, then there is a smallest monotone class containing $\mathscr{A}$, which is called the* monotone class generated by $\mathscr{A}$.

Proof. Suppose $\mathscr{X}$ is a σ-algebra on a set X. Then each monotone increasing sequence of subsets of X in $\mathscr{X}$, then clearly the union of the sequence is a countable union and consequently contained in $\mathscr{X}$. The other case is handled by De Morgan's laws.

Suppose $\mathscr{A}$ is a non-empty collection of subsets of X. Notice the collection containing all subsets of X contains $\mathscr{A}$ and is a monotone class. Let $\mathscr{M}$ be the family containing all monotone classes which contain $\mathscr{A}$. We need only show $M = \bigcap \mathscr{M}$ is a monotone class. This is rather obvious. □

Proposition 2.27. *If $\mathscr{A}$ is a non-empty collection of subsets of X, then the σ-algebra $\mathscr{S}$ generated by $\mathscr{A}$ contains the monotone class $\mathscr{M}$ generated by $\mathscr{A}$. Moreover, the inclusion $\mathscr{A} \subseteq \mathscr{M} \subseteq \mathscr{S}$ may be proper.*

Proof. Since $\mathscr{S}$ is closed under countable unions and closed under countable intersections, it follows that $\mathscr{S}$ is a monotone class containing $\mathscr{A}$. Hence, $\mathscr{S}$ contains $\mathscr{M}$.

Consider $X = \mathbb{N}$ with $\mathscr{A} = \{[n] \mid n \in \mathbb{N}\}$. Then $\mathscr{M} = \mathscr{A} \cup \{X\}$ contains X, yet $\mathscr{A}$ does not contain X. Also, $\mathscr{S}$ contains $X \setminus \{1\}$, which is not a member of $\mathscr{M}$. □

Proposition 2.28. *Let f be a complex-valued function defined on a measurable space $(X, \mathscr{X})$. Then f is $\mathscr{X}$-measurable if and only if*

$$A = \{x \in X \mid a < \mathrm{Re} f(x) < b, c < \mathrm{Im} f(x) < d\}$$

belongs to $\mathscr{X}$ for all real numbers a, b, c, d. More generally, f is $\mathscr{X}$-measurable if and only if $f^{-1}(G) \in \mathscr{X}$ for every open set G in the complex plane $\mathbb{C}$.

Proof. Suppose $f = f_1 + if_2$ is $\mathscr{X}$-measurable. Then $B = \{x \in X \mid a < f_1(x) < b\}$ and $C = \{x \in X \mid c < f_2(x) < d\}$ both belong to $\mathscr{X}$. It follows that $A = B \cap C$ belongs to $\mathscr{X}$, giving us the desired result. If on the other hand A belongs to $\mathscr{X}$, then taking $c = -n$, $d = n$ as $n \to \infty$, it follows that B belongs to $\mathscr{X}$. Similarly C belongs to $\mathscr{X}$.

Suppose f is $\mathscr{X}$-measurable, and G is an open set in $\mathbb{C}$. Then for each $x + iy \in G$, there is an open ball $B \subseteq G$ containing $x + iy$ of some radius r. So, $\sqrt{x^2 + y^2} < r$, implying $-\sqrt{r^2 - y^2} < x < \sqrt{r^2 - y^2}$ and $-\sqrt{r^2 - x^2} < y < \sqrt{r^2 - x^2}$. Clearly, B is in $\mathscr{X}$ and consequently so is G. □

Proposition 2.29. *A function f on X to $\mathbb{R}$ is $\mathscr{X}$-measurable if and only if the set $A_\alpha = \{x \in X \mid f(x) > \alpha\}$ belongs to $\mathscr{X}$ for each rational number α.*

Proof. Suppose A_α belongs to $\mathscr{X}$ for each rational number α, and let β be any real number. We show that A_β belongs to $\mathscr{X}$. Suppose $x \in A_\beta$. Then $f(x) > \beta$, and it is a well known result that there then exists a rational number α_x such that $f(x) > \alpha_x > \beta$. Each A_{α_x} is contained in A_β, and $\bigcup_{x \in A_\beta} A_{\alpha_x} = A_\beta$ is a countable union of sets in $\mathscr{X}$, and consequently $A_\beta \in \mathscr{X}$. □

3. Measures

Definition 3.1. A *measure* is an extended real-valued function μ defined on a σ-algebra $\mathscr{X}$ of subsets of X such that

1. $\mu(\varnothing) = 0$;
2. $\mu(E) \geq 0$ for all $E \in \mathscr{X}$; and
3. μ is *countably additive* in the sense that if (E_n) is any disjoint sequence of sets in $\mathscr{X}$, then
$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

Note. If a measure does not take on the value $+\infty$, we say it is *finite*. More generally, if there exists a sequence (E_n) of sets in $\mathscr{X}$ with $X = \bigcup E_n$ such that $\mu(E_n) < +\infty$ for all n, then we say μ is *σ-finite*.

Example 3.2. If $X = \mathbb{R}$ and $\mathscr{X} = \mathscr{B}$, the Borel algebra, then it will be shown later that there exists a unique measure λ defined on $\mathscr{B}$ which coincides with length on open intervals (that is, $\lambda(E) = b - a$ where $E = (a, b)$). This unique measure is usually called *Lebesgue (or Borel) measure.* It is not a finite measure, but it is σ-finite.

We also have the *counting measure*,

$$\mu(E) = \begin{cases} |E| & \text{if } E \text{ is finite;} \\ \infty & \text{if } E \text{ is infinite.} \end{cases}$$

Also, the *discrete measures*, where

$$\mu(E) = \sum_{x_n \in E} p_n,$$

where (x_n) is a sequence in X and (p_n) a sequence in $[0, \infty)$.

The *Dirac-δ-measure* concentrated at $x_0 \in X$ defined by

$$\delta_{x_0}(E) = \begin{cases} 1 & x_0 \in X; \\ 0 & \text{otherwise.} \end{cases}$$

If I is an interval with endpoints $b < a$, then

(1) $\mu(I) = b - a$ (length of interval);
(2) $E \in \mathscr{S}$ implies $E + x \in \mathscr{S}$ (where $x \in \mathbb{R}$), so $\mu(E + x) = \mu(E)/$

The *Riemann/Lebesgue-Stieltjes measures.* Let $g : \mathbb{R} \to \mathbb{R}$ be increasing and continuous from right. Then the Lebesgue-Stieltjes measure satisfies

$$\mu([a, b]) = g(b) - g(a).$$

For example, $g(x) = x$ yields Lebesgue measure. Also,

$$g(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases}$$

yields the Dirac δ-measure centred at 0.

Proposition 3.3. *For the Lebesgue measure, we cannot have $\mathscr{S} = \mathscr{P}(\mathbb{R})$.*

Proof. Construct a set $E \notin \mathscr{S}$. Define an equivalence relation on $(0, 1) \subset \mathbb{R}$ by $x \sim y$ if $x - y \in \mathbb{Q}$. This partitions $(0, 1)$ into disjoint subsets. By axiom of choice, we can choose an element from each equivalence class to yield E. Observe that

$$(0, 1) \subset S := \bigcup_{r \in \mathbb{Q} \cap [-1,1]} E + r \subset (-1, 2),$$

and $(E + r) \cap (E + s) = \varnothing$ if $r \neq s$. So, by countable additivity,

$$\mu(S) = \sum_{r \in \mathbb{Q} \cap [-1,1]} \mu(E + r) = \sum_{r \in \mathbb{Q} \cap [-1,1]} \mu(E) = \begin{cases} \infty & \text{if } \mu(E) > 0; \\ 0 & \text{if } \mu(E) = 0. \end{cases}$$

Hence,

$$1 = \mu((0, 1)) \leq \mu(S) \leq \mu((-1, 2)) = 3,$$

a contradiction. ☐

Lemma 3.4. *Let μ be a measure defined on a σ-algebra $\mathscr{X}$. If E and F belong to $\mathscr{X}$ and $E \subseteq F$, then $\mu(E) \leq \mu(F)$. If $\mu(E) < +\infty$, then $\mu(F \setminus E) = \mu(F) - \mu(E)$.*

Proof. As E and $F \setminus E$ are disjoint, it follows that $\mu(F) = \mu(F \setminus E) + \mu(E)$. Since $\mu(F \setminus E) \geq 0$, we get that $\mu(F) \geq \mu(E)$. If $\mu(E) < +\infty$, then $\mu(F \setminus E) = \mu(F) - \mu(E)$. ☐

If $E_1 \subseteq E_2 \subseteq \ldots$, then we say (E_n) is expanding/increasing. If $F_1 \supset F_2 \supset \ldots$, then we say (F_n) is contracting/decreasing.

Lemma 3.5. *Let μ be a measure defined on a σ-algebra $\mathscr{X}$.*

(1) *If (E_n) is an increasing sequence in $\mathscr{X}$, then*

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim \mu(E_n).$$

(2) *If (F_n) is a decreasing sequence in $\mathscr{X}$ and if $\mu(F_1) < +\infty$, then*

$$\mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim \mu(F_n).$$

Proof. If $\mu(E_n) = +\infty$ for some n, then (1) follows trivially. Therefore, we assume $\mu(E_n) < +\infty$ for all n.

Let $A_1 = E_1$ and $A_n = E_n \setminus E_{n-1}$ for each $n > 1$. Then (A_n) is a disjoint sequence of sets in $\mathscr{X}$ such that $E_n = \bigcup_{j=1}^{n} A_j$ and $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n$. Since μ is countably additive,

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(A_n) = \lim \sum_{n=1}^{m} \mu(A_n) = \lim \mu(E_n),$$

where final equality holds as $\mu(A_n) = \mu(E_n) - \mu(E_{n-1})$ for $n > 1$ (proving (1)).

Let $E_n = F_1 \setminus F_n$, so that (E_n) is an increasing sequence of sets in $\mathscr{X}$. By (1), we get that

$$\begin{aligned} \mu\left(\bigcup_{n=1}^{\infty} E_n\right) &= \mu\left(F_1 \setminus \bigcap_{n=1}^{\infty} F_n\right) \\ &= \mu(F_1) - \mu\left(\bigcap_{n=1}^{\infty} F_n\right) \\ \implies \qquad \mu(F_1) - \mu\left(\bigcap_{n=1}^{\infty} F_n\right) &= \lim \mu(E_n) \\ &= \lim \left(\mu(F_1) - \mu(F_n)\right), \end{aligned}$$

and after rearranging this yields (2). □

Definition 3.6. A *measure space* is a triple $(X, \mathscr{X}, \mu)$ consisting of a set X, a σ-algebra $\mathscr{X}$ of subsets of X, and a measure μ defined on $\mathscr{X}$.

We shall say that a certain proposition holds *μ-almost everywhere* if there exists a subset $N \in \mathscr{X}$ with $\mu(N) = 0$ such that the proposition holds on the complement of N. Thus we say that two functions f, g, are *equal μ-almost every-where* or that they are *equal for μ-almost all x* in case $f(x) = g(x)$ when $x \notin N$, for some $N \in \mathscr{X}$ with $\mu(N) = 0$. In this case we will write $f = g$, μ-a.e.

In similar manner, we say that a sequence (f_n) of functions on X *converges μ-almost everywhere* (or *converges for μ-almost all x* if there exists a set $N \in \mathscr{X}$ with $\mu(N) = 0$ such that $f(x) = \lim f_n(x)$ for $x \notin N$. In this case we often write $f = \lim f_n$, μ-a.e.

Let P be a statement about the points in a set $E \in \mathscr{S}$, and $E_0 = \{x \in E \mid P(x) \text{ is false}\}$. If there exists $F \in \mathscr{S}$, $F \supset E_0$ and $\mu(F) = 0$, we say $P(x)$ is true for μ-almost-every $x \in E$, or P is μ-almost-everywhere.

Note: Lebesgue measure on single point is 0. For

$$\mu(\{a\}) = m([a, b]) - m((a, b]) = (b - a) - (b - a) = 0.$$

Thus, countable subsets of $\mathbb{R}$ have zero Lebesgue measure. Take counting measure, so a statement holding a.e. is exactly same as holding everywhere (only set measure zero is empty set).

Definition 3.7. If $\mathscr{X}$ is a σ-algebra of subsets of a set X, then a real-valued function λ defined on $\mathscr{X}$ is said to be a *charge* in case $\lambda(\varnothing) = 0$ and λ is countably additive.

Proposition 3.8. *If μ is a measure on $\mathscr{X}$ and A is a fixed set in $\mathscr{X}$, then the function λ, defined for $E \in \mathscr{X}$ by $\lambda(E) = \mu(A \cap E)$, is a measure on $\mathscr{X}$.*

Proof. It immediately follows that λ is non-negative, and that $\lambda(\varnothing) = \mu(A \cap \varnothing) = \mu(\varnothing) = 0$. It remains to show λ is countably additive. To this end, suppose (E_n) is a disjoint sequence of subsets in $\mathscr{X}$. Certainly $(E_n \cap A)$ is a disjoint sequence of subsets in $\mathscr{X}$. As μ is countably additive, we get that

$$\mu\left(\bigcup_{n=1}^{\infty} E_n \cap A\right) = \sum_{n=1}^{\infty} \mu(E_n \cap A) = \sum_{n=1}^{\infty} \lambda(E_n).$$

But we also get that

$$\mu\left(\bigcup_{n=1}^{\infty} E_n \cap A\right) = \mu\left(A \cap \bigcup_{n=1}^{\infty} E_n\right) = \lambda\left(\bigcup_{n=1}^{\infty} E_n\right),$$

giving us λ is indeed countably additive. ☐

Proposition 3.9. *If $\mu_1, \ldots, \mu_n$ are measures on $\mathscr{X}$ and $a_1, \ldots, a_n$ are non-negative real numbers, then the function λ, defined for $E \in \mathscr{X}$ by*

$$\lambda(E) = \sum_{j=1}^{n} a_j \mu_j(E)$$

is a measure on $\mathscr{X}$.

Proof. We need only show λ is countably additive, as other properties trivial. Suppose (E_k) is a disjoint sequence of sets in $\mathscr{X}$. Then, for each $j = 1, \ldots, n$, we have

$$a_j \mu_j\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} a_j \mu_j(E_k).$$

Hence,

$$\begin{aligned}\lambda\left(\bigcup_{k=1}^{\infty} E_k\right) &= \sum_{j=1}^{n} a_j \mu_j\left(\bigcup_{k=1}^{\infty} E_k\right) \\ &= \sum_{j=1}^{n}\left(\sum_{k=1}^{\infty} a_j \mu_j(E_k)\right) \\ &= \sum_{k=1}^{\infty}\left(\sum_{j=1}^{n} a_j \mu_j(E_k)\right) \\ &= \sum_{k=1}^{\infty} \lambda(E_k),\end{aligned}$$

and thus λ is indeed countably additive. ☐

Proposition 3.10. *If (μ_n) is a sequence of measures on $\mathscr{X}$ with $\mu_n(X) = 1$ and if λ is defined by*

$$\lambda(E) = \sum_{n=1}^{\infty} 2^{-n}\mu_n(E),$$

then λ is a measure on $\mathscr{X}$ and $\lambda(X) = 1$.

Proof. Similar proof to previous proposition. □

Proposition 3.11. *Let X be an uncountable set and let $\mathscr{X}$ be the collection of all subsets of X. Define μ on E in $\mathscr{X}$ by requiring that $\mu(E) = 0$, if E is countable, and $\mu(E) = +\infty$, if E is uncountable. Then μ is a measure on $\mathscr{X}$.*

Proof. Certainly $\mu(\varnothing) = 0$, and μ is non-negative. It remains to show that μ is countably additive. To this end, suppose (E_n) is a disjoint sequence of sets in $\mathscr{X}$. If E_n is countable for all n, then $\bigcup_{n=1}^{\infty} E_n$ is countable. In this case,

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = 0 = \sum_{n=1}^{\infty} \mu(E_n).$$

If on the other hand E_n is uncountable for some n, then it is clear $\bigcup_{n=1}^{\infty} E_n$ is uncountable and equality similarly holds by both equalling $+\infty$. □

Proposition 3.12. *Let $X = \mathbb{N}$ and let $\mathscr{X}$ be the collection of all subsets of $\mathscr{N}$. If E is finite, let $\mu(E) = 0$; if E is infinite, let $\mu(E) = +\infty$. Then μ is not countably additive, and thus not a measure on $\mathscr{X}$.*

Proof. Consider (E_n) defined by $E_n = \{n\}$ for each $n \in \mathbb{N}$. Then (E_n) is a disjoint sequence in $\mathscr{X}$, with

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu(\mathbb{N}) = +\infty$$

and

$$\sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} 0 = 0.$$

Thus, μ is not countably additive. □

Proposition 3.13. *Let $X = \mathbb{N}$ and let $\mathscr{X}$ be the σ-algebra of all subsets of $\mathbb{N}$. If (a_n) is a sequence of non-negative real numbers and if we define μ by $\mu(\varnothing) = 0$ and $\mu(E) = \sum_{n\in E} a_n$, $E \neq \varnothing$, then μ is a measure on $\mathscr{X}$. Conversely, every measure on $\mathscr{X}$ is obtained in this way for some sequence (a_n) in $\overline{\mathbb{R}}^{+}$.*

Proof. We firstly show μ is a measure by showing it is countably additive. To this end, suppose (E_n) is a disjoint sequence in $\mathscr{X}$. Let $E = \bigcup_{n=1}^{\infty} E_n$. Observe that

$$\mu(E) = \sum_{m\in E} a_m = \sum_{n=1}^{\infty}\left(\sum_{m\in E_n} a_m\right) = \sum_{n=1}^{\infty} \mu(E_n),$$

giving us desired result.

Converse is trivial. □

Example 3.14. We provide an example of a decreasing sequence (F_n) in $\mathscr{X}$ where $\mu(F_1) = +\infty$ and $\mu(\bigcap_{n=1}^{\infty} F_n) \neq \lim \mu(F_n)$. Consider $X = \mathbb{N}$ and $\mu(\{n\}) = n$. The decreasing sequence $F_0 = X$, $F_1 = X \setminus \{1\}$, $F_2 = X \setminus \{1, 2\}$, etc., has $\mu(\bigcap_{n=1}^{\infty} F_n) = 0$, but $\lim \mu(F_n) = +\infty$.

Proposition 3.15. *Let $(X, \mathscr{X}, \mu)$ be a measure space and let (E_n) be a sequence in $\mathscr{X}$. Then*

$$\mu(\liminf E_n) \leq \liminf \mu(E_n).$$

Proof. Firstly, observe that

$$\liminf E_n = \bigcup_{m=1}^{\infty}\left(\bigcap_{n=m}^{\infty} E_n\right)$$

and

$$\liminf \mu(E_n) = \sup_m \inf_{n\geq m} \mu(E_n).$$

Since $(\bigcap_{n=m}^{\infty} E_n)$ is a monotone increasing sequence in $\mathscr{X}$, it follows that

$$\mu(\liminf E_n) = \lim \mu\left(\bigcap_{n=m}^{\infty} E_n\right).$$

Fixing m, we get that $\bigcap_{n=m}^{\infty} E_n \subseteq E_k$ for all $k \geq m$. Hence,

$$\mu\left(\bigcap_{n=m}^{\infty} E_n\right) \leq \mu(E_k)$$

for all $k \geq m$. It follows that

$$\begin{aligned} \mu\left(\bigcap_{n=m}^{\infty} E_n\right) &\leq \inf_{n\geq m} \mu(E_n) \\ &\leq \sup_m \inf_{n\geq m} \mu(E_n) \\ &= \liminf \mu(E_n) \\ \implies \qquad \mu(\liminf E_n) &\leq \liminf \mu(E_n), \end{aligned}$$

as desired. □

Proposition 3.16. *Let $(X, \mathscr{X}, \mu)$ be a measure space and let (E_n) be a sequence in $\mathscr{X}$. Then*

$$\limsup \mu(E_n) \leq \mu(\limsup E_n)$$

when $\mu(\bigcup_{n=1}^{\infty} E_n) < +\infty$.

Proof. Firstly, observe that

$$\limsup E_n = \bigcap_{m=1}^{\infty}\left(\bigcup_{n=m}^{\infty} E_n\right)$$

and

$$\limsup \mu(E_n) = \inf_m \sup_{n\geq m} \mu(E_n).$$

Since $(\bigcup_{n=m}^{\infty} E_n)$ is a monotone decreasing sequence, when we assume

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) < +\infty,$$

it follows that

$$\mu(\limsup E_n) = \lim \mu\left(\bigcup_{n=m}^{\infty} E_n\right).$$

One then applies a similar approach to previous proposition.

We now provide an example for when $\mu(\bigcup_{n=1}^{\infty} E_n) = +\infty$, and the inequality fails. Take $X = \mathbb{N}$ with $\mathscr{X}$ to be the power set of X, and $\mu(\{n\}) = n$ for all $n \in X$. Also take $E_n = \{n\}$ for all n. Then $\mu(\bigcup_{n=1}^{\infty} E_n) = +\infty$. Since $\limsup E_n = \varnothing$, we know that $\mu(\limsup E_n) = 0$. Yet, $\limsup \mu(E_n) = +\infty$. □

Proposition 3.17. *Let $(X, \mathscr{X}, \mu)$ be a measure space and $\mathscr{A} = \{E \in \mathscr{X} \mid \mu(E) = 0\}$. Then*

(1) *$\mathscr{A}$ is not necessarily a σ-algebra;*
(2) *if $E \in \mathscr{A}$ and $F \in \mathscr{X}$, then $E \cap F \in \mathscr{A}$;*

(3) *if E_n belongs to $\mathscr{A}$ for $n \in \mathbb{Z}$, then $\bigcup E_n \in \mathscr{A}$.*

Proof. Since $\mu(X)$ is not necessarily 0 and any σ-algebra on X must contain X, it is easy to see that (1) holds.

Suppose $E \in \mathscr{A}$ and $F \in \mathscr{X}$. Then $E \cap F \subseteq E$, implying $\mu(E \cap F) \leq \mu(E) = 0$, giving us that $\mu(E \cap F) = 0$ as μ is non-negative. Thus, $E \cap F \in \mathscr{A}$, proving (2).

Now, suppose E_n belongs to $\mathscr{A}$ for all $n \in \mathbb{Z}$. One can then consider the monotone increasing sequence $(\bigcup_{n=1}^{m} E_n)$. We have that $\mu(\bigcup_{n=1}^{\infty} E_n) = \lim \mu(\bigcup_{n=1}^{m} E_n)$. So we need only show that $A \cup B \in \mathscr{A}$ if $A, B \in \mathscr{A}$. We have that $A \subseteq A \cup B$ with $\mu(A) = 0$, implying that $\mu((A \cup B) \setminus A) = \mu(A \cup B) - \mu(A) = \mu(A \cup B)$. Also, $(A \cup B) \setminus A = B \setminus A$. Hence, $\mu((A \cup B) \setminus A) = \mu(B) - \mu(A) = 0$. Thus, $\mu(A \cup B) = 0$. □

Proposition 3.18. *Let $(X, \mathscr{X}, \mu)$ be a measure space and $\mathscr{A} = \{E \in \mathscr{X} \mid \mu(E) = 0\}$, and let $\mathscr{X}' = \{(E \cup A_1) \setminus A_2 \mid E \in \mathscr{X}\}$ where A_1 and A_2 are arbitrary subsets of sets belonging to $\mathscr{A}$. Then $E' \in \mathscr{X}'$ if and only if it has the form $E \cup A$ where $E \in \mathscr{X}$ and $A \in \mathscr{A}$.*

Note. The σ-algebra $\mathscr{X}'$ is called the *completion* of $\mathscr{X}$ (with respect to μ).

Proof. ($\Longrightarrow$) Suppose $E' \in \mathscr{X}'$. Then $E' = (E \cup A_1) \setminus A_2$ for some $E \in \mathscr{X}$ and $A_1, A_2 \in \mathscr{A}$. We have that

$$(E \cup A_1) \setminus A_2 = (E \cup A_1) \cap X \setminus A_2 = (E \cap X \setminus A_2) \cup (A_1 \cap X \setminus A_2),$$

where $E \cap X \setminus A_2 \in \mathscr{X}$ and $A_1 \cap X \setminus A_2 \in \mathscr{A}$.

($\Longleftarrow$) Choosing $A_1 = A$ and $A_2 = \varnothing$ gives us $E \cup A \in \mathscr{X}'$. □

Proposition 3.19. *With respect to the notation of the previous proposition, let μ' be defined on X' by*

$$\mu'(E \cup A) = \mu(E),$$

when $E \in \mathscr{X}$ and $A \in \mathscr{A}$. Then μ' is well-defined and is a measure on $\mathscr{X}'$ which agrees with μ on $\mathscr{X}$.

Note. The measure μ' is called the *completion* of μ.

Proof. Suppose that $E \cup A = E' \cup A'$. Then $(E \cup A) \setminus E = A \cap X \setminus E$ and $(E' \cup A') \setminus E' = A' \cap X \setminus E'$. It follows that $0 = \mu(A \cap X \setminus E) = \mu(E \cup A) - \mu(E)$ and $0 = \mu(A' \cap X \setminus E') = \mu(E' \cup A') - \mu(E')$. Thus, $\mu(E) = \mu(E')$. □

Proposition 3.20. *Let $(X, \mathscr{X}, \mu)$ be a measure space and let $(X, \mathscr{X}', \mu')$ be its completion. Suppose f is an $\mathscr{X}'$-measurable function on X to $\overline{\mathbb{R}}$. Then there exists an $\mathscr{X}$-measurable function g on X to $\overline{\mathbb{R}}$ which is μ-almost everywhere equal to f.*

Proof. □

Proposition 3.21. *If λ denotes the Lebesgue measure and E is an open subset of $\mathbb{R}$, then $\lambda(E) > 0$ if and only if E is non-empty. If K is a compact subset of $\mathbb{R}$, then $\lambda(K) < +\infty$.*

Proof. Suppose E is non-empty. Then E can be written as the disjoint union of open intervals. Since λ is countably additive, it is clear $\lambda(E) > 0$. The converse is trivial, of course.

Suppose K is a compact subset of $\mathbb{R}$. Then K is closed and bounded. Hence, $K \subseteq [a, b]$ for some $a, b \in \mathbb{R}$ and consequently $\mu(K) \leq \mu([a, b]) = b - a < +\infty$. □

Proposition 3.22. *The Cantor set has Lebesgue measure zero.*

Proof. One can take the nested intersection of compact subsets of $[0, 1]$ to obtain the Cantor set. This gives us a decreasing monotone sequence of subsets of $[0, 1]$, each of which have finite measure, and has the limit of the measure tend towards 0. □

4. The Integral

Definition 4.1. A real-valued function is *simple* if it has only a finite number of values.

A simple measurable function φ can be represented in the form

$$\varphi = \sum_{j=1}^{n} a_j \chi_{E_j},$$

where $a_j \in \mathbb{R}$ and χ_{E_j} is the characteristic function of a set E_j in $\mathscr{X}$. Among these representations for φ there is a unique *standard representation* characterized by the fact that the a_j are distinct and the E_j are disjoint non-empty subsets of X and are such that $X = \bigcup_{j=1}^{n} E_j$.

Definition 4.2. For $f : E \to \overline{\mathbb{R}}$, the *positive part* of f is $f^+ : E \to \overline{\mathbb{R}}^{\geq 0}$ such that

$$f^+(x) = \begin{cases} f^+(x) = f(x) & \text{if } f(x) \geq 0; \\ f^+(x) = 0 & \text{otherwise.} \end{cases}$$

Similarly, we define the *negative part* of f as $f^- : E \to \overline{\mathbb{R}}^{\geq 0}$ such that

$$f^-(x) = \begin{cases} f^-(x) = -f(x) & \text{if } f(x) \leq 0; \\ f^-(x) = 0 & \text{otherwise.} \end{cases}$$

Hence, $f = f^+ - f^-$, and we also have

$$f^+ = \frac{f + |f|}{2} \quad \text{and} \quad f^- = \frac{|f| - f}{2}.$$

Hence, if f is measurable, then so are f^+ and f^-, and converse holds.

Theorem 4.3. (Approximation by simple functions). *Let $(X, \mathscr{S})$ be a measurable space. If f is non-negative (extended real value) $\mathscr{S}$-measurable function of $E \in \mathscr{S}$, then there is an increasing sequence of non-negative $\mathscr{S}$-measurable simple functions f_n on E such that*

$$\lim_{n\to\infty} f_n = f.$$

Proof. Define

$$f_n(x) = \begin{cases} \frac{k}{2^n} & \text{if } \frac{k}{2^n} \leq f(x) \leq \frac{k+1}{2^n} \text{ where } k = 0, \dots, n2^n - 1; \\ n & \text{if } f(x) \geq n. \end{cases}$$

Hence,

$$f_n(x) = n\chi_{B_n} + \sum_{k=0}^{n2^n-1} \frac{k}{2^n}\chi_{A_k}$$

where $A_k = f^{-1}\big([\frac{k}{2^n}, \frac{k+1}{2^n}]\big)$ and $B_n = f^{-1}([n, \infty])$. Hence, f_n is $\mathscr{S}$-measurable and clearly increasing. Moreover,

$$0 \leq f(x) - f_n(x) \leq \frac{1}{2^n} \to 0,$$

giving us f is the limit of f_n. □

Fix a measure $(X, \mathscr{S}, \mu)$. Let $M^+(X, \mathscr{S}) = M^+$ be the set of non-negative $\mathscr{S}$-measurable functions on X.

Definition 4.4. If $u \in M^+$ is simple, with standard representation

$$u = \sum_{i=1}^{n} \alpha_k \chi_{A_k}$$

then we define the *integral* of u over X with respect to μ as

$$\int_X u d\mu = \sum_{k=1}^{n} \alpha_k \mu(A_k) \in [0, \infty],$$

where $0 \cdot \infty = 0$.

If $f \in M^+(X, \mathscr{S})$ (non-negative, measurable), then

$$\int_X f d\mu = \sup \left\{ \int_X u d\mu \;\middle|\; 0 \leq u \leq f,\ u \text{ is simple and measurable} \right\}.$$

If $E \in \mathscr{S}$, then the integral of f on E is

$$\int_E f d\mu = \int_X f \chi_E d\mu.$$

Definition 4.5. If φ is a simple function in $M^+(X, \mathscr{X})$ with the standard representation, we define the *integral* of φ with respect to μ to be the extended real number

$$\int \varphi d\mu = \sum_{j=1}^{n} a_j \mu(E_j).$$

Lemma 4.6. (1) *If φ and ψ are simple functions in $M^+(X, \mathscr{X})$ and $c \geq 0$, then*

$$\int c\varphi d\mu = c \int \varphi d\mu,$$
$$\int (\varphi + \psi) d\mu = \int \varphi d\mu + \int \psi d\mu.$$

(2) *If λ is defined for E in $\mathscr{X}$ by*

$$\lambda(E) = \int \varphi \chi_E d\mu,$$

then λ is a measure on $\mathscr{X}$.

Proof. The first property is obvious; for the second suppose $\varphi = \sum_{j=1}^{n} a_j \chi_{E_j}$ and $\psi = \sum_{j=1}^{m} b_j \chi_{F_j}$.

For (2),

$$\lambda(E) = \int_X u \chi_E d\mu = \int_X \left(\sum_j \alpha_j \chi_{E \cap A_j} \right) d\mu = \sum_j \alpha_j \left(\int_X \chi_{E \cap A_j} d\mu \right) = \sum_j \alpha_j \mu_j(E).$$

by linearly and $\mu_j(E) = \mu(E \cap A_j)$. □

Lemma 4.7. *Let $f, g \in M^+(X, \mathscr{S})$ and $E, F \in \mathscr{S}$. Then*

(1) *If $f \leq g$ on E, then $\int_E f d\mu \leq \int_E g d\mu$.*
(2) *If $E \subseteq F$, then*

$$\int_E f d\mu \leq \int_F f d\mu.$$

Proof. Part (1) follows trivially.

Part (2) follows from $f\chi_E \leq f\chi_F$ and applying part (1). □

Theorem 4.8. (Monotone Convergence). *If (f_n) is an increasing sequence of non-negative measurable functions, then*

$$\lim \int_X f_n d\mu = \int_X \lim f_n d\mu.$$

Proof. Since (f_n) is increasing it has a limit (in the extended sense) such that $f = \lim f_n$ which is non-negative and measurable. Note $f_n \leq f$, so by Lemma $\int_X f_n d\mu \leq \int_X f d\mu$ for all n and hence $\lim \int_X f_n d\mu \leq \int_X f d\mu$.

Fix $u \leq f$, simple non-negative measurable function and $0 < \alpha < 1$. Let A_n be the set $\{x \mid f_n(x) \geq \alpha u(x)\}$. Then $A_n \in \mathscr{S}$ and (A_n) is expanding. Also, $\bigcup A_n = X$. By monotonicity,

$$\int_{A_n} \alpha u d\mu \leq \int_{A_n} f_n d\mu \leq \int_X f_n d\mu \leq \overbrace{\lim \int_X f_n d\mu}^{\text{constant}}.$$

Recall, $\lambda : E \mapsto \int_E u d\mu$ is a measure. Since (A_n) is expanding, $\lim \lambda(A_n) = \lambda(\bigcup A_n)$. Thus,

$$\lambda(X) = \lambda\Big(\bigcup A_x\Big) = \int_X \alpha u d\mu \leq \lim \int_X f_n d\mu.$$

By linearity,

$$\begin{aligned} & \alpha \int_X u d\mu \leq \lim \int_X f_n d\mu \\ \implies \quad & \int_X u d\mu \leq \lim \int_X f_n d\mu, \quad \text{by } \alpha \to 1, \\ \implies \quad & \int_X f d\mu \leq \lim \int_X f_n d\mu. \end{aligned}$$

□

Note. There is no corresponding result for Riemann integration. For example, let (a_n) be the rationals in $[a, b]$ arranged as a sequence. Let $f_n = \chi_{\{q_1,\ldots,q_n\}}$ is Riemann integrable and is increasing. Also, has a limit $f = \chi_{[a,b]\cap\mathbb{Q}}$, but this limit is not Riemann integrable. Of course, it is Lebesgue integrable.

Theorem 4.9. *Let $f, g \in M^+(X, \mathscr{S})$ and $\alpha, \beta \in [0, \infty)$. Then*

$$\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu.$$

Proof. There exists increasing sequences (u_n) and (v_n) of non-negative simple measurable functions with

$$\lim_{n\to\infty} u_n = f \text{ and } \lim_{n\to\infty} v_n = g.$$

Observe that $(\alpha u_n + \beta v_n)$ is non-negative, increasing, simple measurable functions with limit $\alpha f + \beta g$. By the monotone convergence theorem,

$$\begin{aligned} \int_X (\alpha f + \beta g) d\mu &= \int_X \lim (\alpha u_n + \beta v_n) d\mu \\ &= \lim \int_X (\alpha u_n + \beta v_n) d\mu \\ &= \lim \left(\alpha \int_X u_n d\mu + \beta \int_X v_n d\mu \right), \end{aligned}$$

where final equality holds by linearity. By monotonicity, $n \mapsto \int_X u_n d\mu$ is increasing and hence has a limit in the extended sense. Thus,

$$\begin{aligned}\int_X (\alpha f + \beta g) d\mu &= \alpha \lim \int_X u_n d\mu + \beta \lim \int_X v_n d\mu \\ &= \alpha \int_X f d\mu + \beta \int_X g d\mu.\end{aligned}$$

□

Theorem 4.10. *Let f be a non-negative measurable function on X, and define*

$$\nu(E) := \int_E f d\mu \ E \in \mathscr{S}.$$

Then ν is a measure on $\mathscr{S}$.

Proof. $(M1)$ holds:

$$\nu(\varnothing) = \int_\varnothing f d\mu = \int_X f \chi_\varnothing d\mu = \int_X 0 d\mu = 0.$$

$(M2)$ holds: $f \geq 0$ so by monotonicity of the integral, $\int_E f d\mu \geq \int_E 0 d\mu = 0$. One could also use $(M1)$ and monotonicity, observing E contains $\varnothing$.

Now, $(M3)$: Let (E_n) be a disjoint sequence of measurable sets with union E. Then, since $\chi_E = \sum_{j=1}^\infty \chi_{E_j} = \lim \sum_{j=1}^n \chi_{E_j}$,

$$\begin{aligned}\nu(E) &= \int_E f d\mu = \int_X f \chi_E d\mu = \int_X \sum_{j=1}^\infty \chi_{E_j} f d\mu \\ &= \int_X \lim \sum_{j=1}^n \chi_{E_j} f d\mu = \lim \int_X \sum_{j=1}^n \chi_{E_j} f d\mu = \lim \sum_{j=1}^n \int_X \chi_{E_j} f d\mu \\ &= \sum_{j=1}^\infty \int_X \chi_{E_j} f d\mu = \sum_{j=1}^\infty \int_{E_j} f d\mu = \sum_{j=1}^\infty \nu(E_j).\end{aligned}$$

□

Lemma 4.11. (Fatou's Lemma). *If (f_n) is a sequence of non-negative measurable functions on X, then*

$$\int_X \liminf f_n d\mu \leq \liminf \int_X f_n d\mu.$$

Proof. Let $h_n = \inf_{k \geq n} f_n$, which is an increasing sequence in $M^+(X, \mathscr{S})$ with limit $\liminf f_n$. By monotone convergence,

$$\int_X \liminf f_n d\mu = \int_X \lim h_n d\mu = \lim \int_X h_n d\mu.$$

Also, $h_n \leq f_n$ implying $\int_X h_n d\mu \int_X f_n d\mu$ and hence

$$\liminf \int_X h_n d\mu \leq \liminf \int_X f_n d\mu.$$

Moreover, $\liminf \int_X h_n d\mu = \lim \int_X h_n d\mu$ and we are done. □

Example 4.12. The inequality for Fatou's Lemma can be strict. Let μ be the Lebesgue measure, and $f_n = \frac{1}{n}\chi_{[0,n]}$ which has integral value 1. Then

$$\int_X \lim f_n d\mu = \int_X 0 d\mu = 0 < \lim \int_X f_n d\mu = 1.$$

Theorem 4.13. *Let f be non-negative, measurable on X. Then $f = 0$ μ-a.e. if and only if $\int_X f d\mu = 0$.*

Proof. ($\Longrightarrow$) Suppose $f = 0$ μ-a.e., i.e., $E = \{x \in X \mid f(x) > 0\}$ has $\mu(E) = 0$. Let $f_n = n\chi_E$, where $\lim f_n = \infty\mu_E \geq f$. By Fatou's Lemma and monotonicity,

$$0 \leq \int_X f d\mu \leq \int_X \lim f_n d\mu \leq \liminf \int f_n d\mu = 0.$$

($\Longleftarrow$) Suppose $\int_X f d\mu = 0$. Let $E_n = \left\{x \in X \mid f(x) > \frac{1}{n}\right\}$. Then $f_n = \frac{1}{n}\chi_{E_n}$ has limit 0, and also $f \geq f_n$ on E_n. So, we get that $\frac{1}{n}\mu(E_n) = \int_X f_n \chi_{E_n} \leq \int_X f d\mu = 0$ by monotonicity, and hence $\mu(E_n) = 0$ for all n. Thus, $\mu(E) = \mu(\bigcup E_n) \leq \sum \mu(E_n) = 0$ and result follows. □

Theorem 4.14. *Let $f, g \in M^+(X, \mathscr{S})$. Then*

(1) *If $f \leq g$ μ-a.e., then*

$$\int_X f d\mu \leq \int_X g d\mu.$$

(2) *If $f = g$ μ-a.e., then*

$$\int_X f d\mu = \int_X g d\mu.$$

Proof. Since $f \leq g$ μ-a.e., $E = \{x \in X \mid f(x) > g(x)\}$ has $\mu(E) = 0$. Now, by linearity it follows

$$\begin{aligned}\int_X f d\mu &= \int_E f d\mu + \int_{X \setminus E} f d\mu \\ &\leq \int_E g d\mu + \int_{X \setminus E} g d\mu = \int_X g d\mu.\end{aligned}$$

(2) follows from (1). □

5. Integrable Functions

Fix $(X, \mathscr{S}, \mu)$. The space $L = L(\mu)$ of all *integrable functions* consists of all extended-real valued measurable functions on X whose positive and negative parts f^+, f^- have finite integrals over X with respect to μ. For $f \in L$, we define the *integral of f over X w.r.t.* μ by

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu \in \mathbb{R}.$$

A non-example is $f(x) = \frac{1}{x}$ on $(0, 1]$, which has Lebesgue integral $\int_{\mathbb{R}} \frac{1}{x} dx = \infty$ but it is not Lebesgue integrable.

Theorem 5.1. *If f is measurable, then $f \in L$ iff $|f| \in L$, i.e., $\int_X |f| d\mu < \infty$, in this case $\left|\int_X f d\mu\right| \leq \int_X |f| d\mu$.*

Proof. ($\Longrightarrow$) Suppose $f \in L$, i.e., $\int f^+, \int f^- < \infty$. Recall that $|f| = f^+ + f^-$. Hence, $\int |f| = \int f^+ + f^- = \int f^+ + \int f^- < \infty$ using linearity, and result follows.

($\Longleftarrow$) Suppose $|f| \in L$, i.e., $\int |f| < \infty$. Recall $0 \leq f^+, f^- \leq |f|$. By monotonicity of the integral $\int f^+, \int f^- \leq \int |f| < \infty$ so finite implying $f \in L$.

Also,

$$\begin{aligned}\left|\int_X f d\mu\right| &= \left|\int_X f^+ d\mu - \int_X f^- d\mu\right| \\ &\leq \left|\int_X f^+ d\mu\right| + \left|\int_X f^- d\mu\right| \\ &= \int_X f^+ d\mu + \int_X f^- d\mu \\ &= \int_X (f^+ + f^-) d\mu \\ &= \int_X |f| d\mu.\end{aligned}$$

□

Lemma 5.2. *Suppose $f \in L$. Then*

(1) *f is finite-valued a.e. on X.*

(2) *If*

$$\tilde{f}(x) := \begin{cases} f(x), & \text{if } f(x \in \mathbb{R}); \\ 0, & f(x) \in \{\pm\infty\}. \end{cases}$$

Then $\tilde{f} = f$ μ-a.e. and $\int_X \tilde{f} d\mu = \int_X f d\mu$.

Proof. Let $E = \{x \in X \mid |f(x)| = \infty\}$. Now,

$$\infty > \int_X |f| d\mu \geq \int_E |f| d\mu = \int_E \infty d\mu \geq \int_E n d\mu = n\mu(E)$$

for all n. Hence, $\mu(E) = 0$, proving (1).

Now, on E $\tilde{f} = 0$, and on $X \setminus E$ $\tilde{f} = f$, giving us $\tilde{f}$ is measurable and equal to f μ-a.e. Also, $\left(\tilde{f}\right)^+ = f^+$ μ-a.e. and $\left(\tilde{f}\right)^- = f^-$ μ-a.e., giving us

$$\int_X \tilde{f} d\mu = \int_X \left(\tilde{f}\right)^+ d\mu - \int_X \left(\tilde{f}\right)^- d\mu = \int_X f^+ d\mu - \int_X f^- d\mu = \int_X f d\mu.$$

□

Theorem 5.3. *Let $f, g \in L$ and $\alpha \in \mathbb{R}$. Then $f + g$, $\alpha f \in L$, $\int_X f + g d\mu = \int_X f d\mu + \int_X g d\mu$, and $\int_X \alpha f d\mu = \alpha \int_X f d\mu$.*

Proof. Observe

$$(\alpha f)^+ = \begin{cases} \alpha f^+ & \text{if } \alpha \geq 0; \\ -\alpha f^- & \text{if } \alpha < 0; \end{cases}$$

and

$$(\alpha f)^- = \begin{cases} \alpha f^- & \text{if } \alpha \geq 0; \\ -\alpha f^+ & \text{if } \alpha < 0. \end{cases}$$

Now, $\alpha \geq 0$ implies

$$\begin{aligned}\int \alpha f &= \int \alpha f^+ - \int \alpha f^- \\ &= \alpha \int f^+ - \alpha \int f^- \\ &= \alpha\left(\int f^+ - \int f^-\right) \\ &= \alpha \int f,\end{aligned}$$

and $\alpha < 0$ implies

$$\begin{aligned}\int \alpha f &= \int -\alpha f^- - \int -\alpha f^+ \\ &= \int \alpha f^+ - \int \alpha f^-\end{aligned}$$

and apply previous case.

Note $f+g \in L$, since $\int |f+g| \leq \int |f| + \int |g| < \infty$, so $f+g \in L$. Suppose without loss of generality f and g are finite valued. We get $(f+g) = (f+g)^+ - (f+g)^-$. Hence,

$$(f+g)^+ + f^- + g^- = f^+ + g^+ + (f+g)^-.$$

By linearity for non-negative functions,

$$\int (f+g)^+ + \int f^- + \int g^- = \int f^+ + \int g^+ + \int (f+g)^-.$$

All integrals are finite, so we can rearrange to get $\int f+g = \int f + \int g$. □

Theorem 5.4. (Dominated Convergence Theorem). *Suppose $f_n \in L$ for all n and $f_n \to f$ μ-a.e. and if there exists function $g \in L$ with $|f_n| \leq g$ for all n (μ-a.e.) then $f \in L$ and $\int_X \lim f_n d\mu = \int_X f d\mu = \lim \int_X f_n d\mu$.*

Proof. Let $E = \{x \in X \mid g(x) = \infty\}$, such that $\mu(E) = 0$. On $X \setminus E$, $|f_n|$ is bounded, and so $\lim |f_n| = |\lim f_n| = |f| \leq g$, since $|\cdot|$ is a continuous function. So by monotonicity,

$$\int_X |f| d\mu \leq \int_X g d\mu < \infty \Longrightarrow f \in L.$$

Assume without loss of generality, f_n, f are finite valued. Observe that (a.e.) $f_n + g \geq 0$. By Fatou's Lemma,

$$\begin{aligned}&\int_X \liminf (f_n + g) d\mu \leq \liminf \int_X (f_n + g) d\mu \\ \Longrightarrow \quad &\int_X f + g d\mu \leq \liminf \int_X f_n d\mu + \int_X g d\mu \\ \Longrightarrow \quad &\int_X f d\mu \leq \liminf \int_X f_n d\mu \leq \int_X f_n d\mu.\end{aligned}$$

Note $(-f_n)$ satisfies conditions of the Dominated Convergence Theorem. Hence,

$$\int_X -f d\mu \leq \liminf \int_X -f_n d\mu = \liminf - \int_X f_n d\mu = -\limsup \int_X f_n d\mu,$$

and thus

$$\begin{aligned}&\limsup \int_X f_n d\mu \leq \int_X f d\mu \leq \liminf \int_X f_n d\mu \\ \Longrightarrow \quad &\int_X f d\mu = \lim \int_X f_n d\mu.\end{aligned}$$

□

Theorem 5.5. (1) *If g is measurable on X, and $f \in L$ and $g = f$ μ-a.e., then $g \in L$ with*

$$\int_X f d\mu = \int_X g d\mu.$$

(2) *If $f, g \in L$ and $f \leq g$ μ-a.e., then*

$$\int_X f d\mu \leq \int_X g d\mu.$$

(3) *If $\mu(E) = 0$, and f is measurable on E, then $f\chi_E \in L$ and $\int_E f d\mu = 0$.*

Proof. Let $E = \{x \in X \mid f(x) \neq g(x)\}$, $\mu(E) = 0$. Now,

$$\begin{aligned}\int_X |g| d\mu &= \int_{X \setminus E} |g| d\mu + \overbrace{\int_E |g| d\mu}^{=0} \\ &= \int_{X \setminus E} |f| d\mu \\ &\leq \int_X |f| d\mu < \infty, \text{ since } f \in L.\end{aligned}$$

Thus, $g \in L$. Integrate

$$\begin{aligned}\int_X g d\mu &= \int_{X \setminus E} f d\mu + \overbrace{\int_E g d\mu}^{=\int_E g^+ d\mu - \int_E g^- d\mu = 0} \\ &= \int_{X \setminus E} f d\mu + \overbrace{\int_E f d\mu}^{=0} \\ &= \int_X f d\mu,\end{aligned}$$

proving (1).

$f \leq g$ iff $g - f \geq 0$, μ-a.e., implying $\int_X g - f d\mu \geq 0$ by monotonicity, and so $\int_X g d\mu - \int_X f d\mu \geq 0$ by linearity. Thus, $\int_X f d\mu \leq \int_X g d\mu$, proving (2).

(3) follows trivially by definition, since $\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu = 0$. □

6. The Lebesgue Spaces L_p

Lebesgue L_p spaces functions as vectors

$$(f + g)(x) = f(x) + g(x)$$
$$(cf)(x) = cf(x).$$

Measure the size of f via integral, e.g.,

$$\|f\| = \int |f| d\mu.$$

Equivalence class of functions: $f \sim g$ if $f = g$ a.e., and so $[f] = \{g : X \to \mathbb{R} \mid f = g \text{ a.e.}\}$. So our vectors are actually equivalence classes of functions.

Proposition 6.1. *Addition and scalar multiplication is well-defined.*

Proof. We wish to show addition as

$$[f] + [g] = [f + g]$$

is well-defined. Say $f_1 \sim f_2$ and $g_1 \sim g_2$. Want to know whether $f_1 + g_1$ and $f_2 + g_2$ are equivalent. Let $E = \{x \in X \mid f_1(x) + g_1(x) \neq f_2(x) + g_2(x)\}$. We know $f_1(x) = f_2(x)$ a.e., and $g_1(x) = g_2(x)$ a.e., so set $E_1 = \{x \in X \mid f_1(x) \neq f_2(x)\}$ and $E_2 = \{x \in X \mid g_1(x) \neq g_2(x)\}$. Now, $E_1 \subseteq N_1$ and $E_2 \subseteq N_2$ where $\mu(N_1) = 0$ and $\mu(N_2) = 0$. Observe $E \subseteq E_1 \cup E_2 \subseteq N_1 \cup N_2$, so $\mu(e) \leq \mu(N_1 \cup N_2) \leq \mu(N_1) + \mu(N_2) = 0$, and hence addition is well-defined.

Let $E = \{x \in X \mid f_1(x) \neq f_2(x)\}$, so $E \subseteq N$ where $\mu(N) = 0$. If $c = 0$, then $0 = 0$ always. If $c \neq 0$,

$$cf_1(x) \neq cf_2(x) \implies f_1(x) \neq f_2(x),$$

so scalar multiplication is well-defined. □

Definition 6.2. Let $1 \leq p < \infty$. For measure space $(X, \mathscr{S}, \mu)$, define

$$L_p = \left\{ [f] : X \to \mathbb{R} \;\middle|\; \int_X |f|^p d\mu < \infty \right\}.$$

Proposition 6.3. *L_p is a Banach space (complete normed vector space) under*

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}}.$$

Proof. Firstly, $L_p(\mu)$ is a vector subspace of μ-equivalence classes. Let $f, g \in L_p(\mu)$. Then

$$|f+g|^p \leq (|f|+|g|)^p \leq (2\max\{|f|,|g|\}) = 2^p \max\{|f|^p, |g|^p\} \leq 2^p(|f|^p + |g|^p).$$

Hence,

$$\int |f+g|^p \leq 2^p \int \left(\int |f| + \int |g| \right) < \infty,$$

so $f+g \in L_p$. Also, $\int |\alpha f|^p = |\alpha|^p \int |f|^p < \infty$, so $\alpha f \in L_p$. Hence, L_p is a vector space.

Now, $\|f\|_p \geq 0$ since integrating positive function. Also, $\|f\|_p = 0$ if and only if $\int |f|^p = 0$, iff $|f|^p = 0$ μ-a.e. iff $f = 0$ μ-a.e., so $[f] = [0]$. When showing L_p closed under scalar multiplication, easy to see that $\|\alpha f\|_p = |\alpha| \|f\|_p$.

To prove the triangle inequality of L_p, must use Muntowski's inequality, which we prove using Holder's inequality. For $1 < p < \infty$ the conjugate index q is given by $\frac{1}{p} + \frac{1}{q} = 1$. Holder's inequality: $f \in L_p$, $g \in L_q$ we have $fg \in L_1$ and

$$\left| \int fg d\mu \right| \leq \int |fg| d\mu \leq \|f\|_p \|g\|_q.$$

Proof of Holder's Inequality: We use given $A, B > 0$ that $AB \leq \frac{A^p}{p} + \frac{B^q}{q}$. Suppose $\|f\|_p$, $\|g\|_q \neq 0$ and $f, g \geq 0$. Then

$$\int \frac{fg}{\|f\|_p \|g\|_q} d\mu \leq \int \frac{f^p}{p\|f\|_p^p} + \frac{g^q}{q\|g\|_q^q} d\mu = \frac{1}{p} + \frac{1}{q} = 1,$$

and so by rearranging we get the desired inequality.

Now, Muntoskii's inequality; given $f, g \in L_p$ and $1 \leq p < \infty$. Note $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ is easy for $p = 1$ by triangle inequality. For $p > 1$,

$$|f+g|^p = |f+g|^{p-1}|f+g| \leq |f+g|^{p-1}(|f|+|g|).$$

Hence,

$$\int |f+g|^p \leq \int |f+g|^{p-1}(|f|+|g|)$$

giving us $\left\| (f+g)^{p-1} \right\|_q \left(\|f\|_p + \|g\|_p \right)$.

We now show L_p is complete. Let (f_n) be a Cauchy sequence in $L_p(\mu)$, i.e., for all $\epsilon > 0$ there exists positive integer N such that $\|f_n - f_m\|_p < \epsilon$ for all $n, m \geq N$. Observe

$$f_n = (f_n - f_{n-1}) + (f_{n-1} - f_{n-2}) + \ldots + (f_2 - f_1) + f_1$$

for each n. There is a subsequence (g_n) of (f_n) with

$$\|g_{k+1} - g_k\|_p < \frac{1}{2^k}.$$

Let $g(x) := |g_1(x)| + \sum_{k=1}^{\infty} |g_{k+1}(x) - g_k(x)|$. This g is non-negative, measurable, and

$$\begin{aligned}\int_X |g|^p d\mu &\leq \liminf \int_X \left(|g_1(x)| + \sum_{k=1}^{n} |g_{k+1}(x) - g_k(x)| \right)^p d\mu \\ &= \left\| |g_1| + \sum_{k=1}^{n} |g_{k+1} - g_k| \right\|_p^p \\ &\leq \liminf \left(\|g_1\|_p + \sum_{k=1}^{n} \|g_{k+1} - g_k\|_p \right)^p \\ &\leq \liminf \left(\|g_1\|_p + \sum_{k=1}^{n} \frac{1}{2^k} \right) = \|g_1\|_p + 1 < \infty,\end{aligned}$$

and hence $g \in L_p(\mu)$. Now, let $E = \{x \in X \mid |g(x)| < \infty\}$, the set where g converges absolutely. So, $\mu(X \setminus E) = 0$. Let

$$f(x) = \begin{cases} g_1(x) + \sum_k (g_{k+1} - g_k) & \text{if } x \in E; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $|f(x)| \leq g(x)$ on E, so $\int |f|^p \leq \int |g|^p d\mu < \infty$, giving us $f \in L_p(\mu)$. Need to show $\|f_n - f\|_p^p = \int_X |f_n - f|^p d\mu \to 0$ as $n \to \infty$. On E, $f(x) = \lim g_n(x)$; $|g_n(x) - f(x)|^p \to 0$ as $n \to \infty$. Note

$$|g_n(x) - f(x)|^p \leq (|g_n(x)| + |f(x)|)^p \leq 2^p g(x)^p.$$

By Dominated convergence theorem,

$$\lim \|g_n - f\|_p^p = \int \lim |g_n - f|^p d\mu = \int 0 = 0,$$

i.e., $g_n \to f$ in L_p. Recall (g_n) subsequence of Cauchy sequence (f_n), so follows $f_n \to f$ and thus L_p is complete. ☐

Note. $L_p(\mu)$ is a Hilbert space if it satisfies the parallelogram law iff $p = 2$ (μ not on singleton). Polarisation identity: $\langle f, g \rangle = \int_X f g d\mu$. Every Hilbert space is isometric to $\ell_2(A)$ for some set A. $\ell_2(\mathbb{N})$ on basis $(e_j)_{j \in \mathbb{N}}$.

7. Construction of measures

Motivation: the length of the rationals. Recall that we can cover $\mathbb{Q}$ with intervals I_j, and choose length of each interval to be $\frac{\epsilon}{2^j}$ where $\mathbb{Q} \subseteq \bigcup_{j=1}^{\infty} I_j$. Then $\mu(\mathbb{Q}) \leq \sum_{j=1}^{\infty} \mu(I_j) = \epsilon$, implying $\mu(\mathbb{Q}) = 0$.

Some examples: Lebesgue measure, Hausdorff measure, Haar measure (which is used on the circle S^1).

Definition 7.1. An *algebra* is a collection $\mathscr{A}$ of subsets of a set X which is closed under finite unions and closed under complementation.

Example 7.2. Let $\mathscr{A}$ be the collection of all finite unions of intervals of the form $[a, b)$; this is an algebra on $\mathbb{R}$ if we allow for half-rays. For e.g., $\mathbb{R} \setminus [a, b) = (-\infty, a) \cup [b, \infty)$. One could actually take this be an algebra instead on $\mathbb{R} \cup \{-\infty\}$. Note that this is not a σ-algebra, since it is not closed under countable unions. For example, $\mathscr{A}$ does not contain $[a, b]$.

Definition 7.3. Let $\mathscr{A}$ be an algebra on X. A function $\mu_0 : \mathscr{A} \to \overline{\mathbb{R}}$ is a *measure on $\mathscr{A}$* if

(1) $\mu_0(\varnothing) = 0$;

(2) $\mu_0(E) \geq 0$ for each $E \in \mathscr{A}$;
(3) if (E_j) is a disjoint sequence in $\mathscr{A}$ with $\bigcup E_j \in \mathscr{A}$, then

$$\mu_0\left(\sum_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu_0(E_j).$$

For example, $\mathscr{A}$ consisting of finite unions of intervals on $\mathbb{R}$ and μ_0 defined as $\mu_0\left(\bigcup_{j=1}^{n} I_j\right) := \sum_{j=1}^{n} \text{length}(I_j)$. This is a measure on algebra $\mathscr{A}$.

We want to define Lebesgue measure by

$$\mu^*(E) = \inf\left\{ \sum_{k=1}^{\infty} \mu_0(A_k) \;\middle|\; A_k \in \mathscr{A}, E \subseteq \bigcup_{k=1}^{\infty} A_k \right\}.$$

This is not a measure in general, but it what is called an outer measure.

Definition 7.4. An *outer measure* is a function $\mu^* : \mathscr{P}(X) \to \overline{\mathbb{R}}$ where

(1) $\mu^*(\varnothing) = 0$;
(2) $E_1 \subseteq E_2$ implies $\mu^*(E_1) \leq \mu^*(E_2)$;
(3) $\mu^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$, i.e., countable sub-additivity.

The outer measure given by a measure on an algebra, for example, have $\mathbb{R}^2$ (think of squares v.s. rectangles).

Example 7.5. *Hausdorff measure* on metric space (X, ρ), where

$$H_\delta^d := \inf\left\{ \sum_{k=1}^{\infty} (\text{diam} A_k)^d \;\middle|\; E \subseteq \bigcup A_k, \text{diam}(A_k) < \delta \right\}$$

is an outer measure ($\delta > 0$ and $d \in \mathbb{R}^{>0}$ is a dimension). Note H_δ^d increases as δ decreases. We define the *Hausdorff outer measure* by $H^d(E) := \lim_{\delta \to 0} H_\delta^d(E)$.

Definition 7.6. Let μ^* be an outer measure on X. We say $E \subset X$ is *μ^*-measurable* if $\mu^*(Q) = \mu^*(Q \cap E) + \mu^*(Q \setminus E)$ for every $Q \subset X$. Note by subaddivity $\mu^*(Q) \leq \mu^*(Q \cap E) + \mu^*(Q \setminus E)$. So being μ^*-measurable is equivalent to

$$\mu^*(Q) \geq \mu^*(Q \cap E) + \mu^*(Q \setminus E) = \mu^*(Q \cap E) + \mu^*(Q \cap (X \setminus E)).$$

Note then that E is μ^*-measurable iff $X \setminus E$ is μ^*-measurable. If $\mu^*(E) = 0$, then E is μ^*-measurable. For $\mu^*(Q) = \mu^*(E) + \mu^*(Q) \geq \mu^*(Q \cap E) + \mu^*(Q \setminus E)$. This implies E is μ^*-measurable implies $\varnothing$ and X μ^*-measurable.

Theorem 7.7. (Caratheodory). *Let μ^* be an outer measure on X, $\mathscr{S}$ be the collection of all μ^*-measurable subsets. Then $\mathscr{S}$ is a σ-algebra. The restriction of μ^* to $\mathscr{S}$ is a complete measure.*

Proof. Firstly show $\mathscr{S}$ is a σ-algebra. We know $\mathscr{S}$ contains $\varnothing$, X and also closed under complementation. Say $\mathscr{A} = (A_n)$ countable subcollection of $\mathscr{S}$. Then

$$\begin{aligned} \mu^*(Q) &\geq \mu^*(Q \cap A_1) + \mu^*(Q \setminus A_1) \\ &\geq \mu^*(Q \cap A_1) + \mu^*((Q \setminus A_1) \cap A_2) + \mu^*((Q \setminus A_1) \setminus A_2) \\ &\geq \sum_{j=1}^{k} \left(\left(Q \setminus \bigcup_{i=1}^{j-1} A_i \right) \cap A_j \right) + \mu^*\left(Q \setminus \bigcup_{i=1}^{k} A_i \right). \end{aligned}$$

Taking limit as $k \to \infty$, $\mu^*(Q) \geq \sum_{j=1}^{\infty} \left(\left(Q \setminus \bigcup_{i=1}^{j-1} A_i \right) \cap A_j \right) + \mu^*(Q \setminus \bigcup_{i=1}^{\infty} A_i)$. Consider $\bigcup_{i=j}^{\infty} \left(Q \setminus \bigcup_{i=1}^{j-1} A_i \right) \cap A_j \supset \bigcup_{n=1}^{\infty} A_n$, so $\mu^*(Q) \geq \mu^*(Q \cap (\bigcup_{n=1}^{\infty} A_n)) + \mu^*(Q \setminus \bigcup_{n=1}^{\infty} A_n)$, so $\bigcup_{n=1}^{\infty} A_n$ is μ^*-measurable, implying $\mathscr{S}$ is a σ-algebra.

Now show μ^* restricted to $\mathscr{S}$ is a measure. $\mu^* |_{\mathscr{S}} (\varnothing) = \mu^*(\varnothing) = 0$, and $\mu^* \geq 0$ by definition. Let (E_n) be a sequence of disjoint sets in $\mathscr{S}$. Let $Q = \bigcup_{n=1}^{\infty} E_n$. Then

$$\mu^*(Q) \geq \sum_{j=1}^{\infty} \left(\left(Q \setminus \bigcup_{i=1}^{j-1} E_i \right) \cap E_j \right) + \mu^* \left(Q \setminus \bigcup_{i=n}^{\infty} E_n \right),$$

implying $\mu^*(\bigcup_{n=1}^{\infty} E_n) \geq \infty_{j=1}^{\infty} \mu^*(E_j)$. As μ^* is an outer measure, $\mu^*(\bigcup_n E_n) \leq \sum \mu^*(E_j)$. This shows restriction of μ^* to $\mathscr{S}$ is a measure on $\mathscr{S}$.

Remains to show $\mu^* |_{\mathscr{S}}$ is complete. If $F \subset E$ and $\mu^*(E) = 0$, $E \in \mathscr{S}$ implies $0 \leq \mu^*(F) \leq \mu^*(E) = 0$ implies $\mu^*(F) = 0$. This implies F is μ^*-measurable i.e., $F \in \mathscr{S}$. □

Theorem 7.8. *Let μ_0 be a measure on an algebra of subsets of X. Then $\mu^*(E) := \inf \{\sum_{k=1}^{\infty} \mu_0(A_k) \mid A_k \in \mathscr{A}, \bigcup A_k \supset E\}$ is an outer measure on X. Further, $\mu^*(A) = \mu_0(A)$ for all $A \in \mathscr{A}$, and every set in $\mathscr{A}$ is μ^*-measurable.*

Proof. μ^* is an outer measure. Since $\varnothing \subset A_k$, $A_k = \varnothing$ implies $\mu^*(\varnothing) = 0$. Also, if $E_1 \subset E_2$ implies $\mu^*(E_1) \leq \mu^*(E_2)$ by definition of infimum. Now, let (E_n) be a sequence of subsets in X. Let $\epsilon > 0$. Choose $A_{nk} \in \mathscr{A}$ such that $\bigcup_{k=1}^{\infty} A_{nk} \supset E_n$, $\sum_{k=1}^{\infty} \mu_0(A_{nk}) < \mu^*(E_n) + \frac{\epsilon}{2^n}$. Then $\bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_{nk}$,

$$\mu^* \left(\bigcup_{n=1}^{\infty} E_n \right) \leq \sum_n \left(\sum_k \mu_0(A_{nk}) \right) < \sum_n \left(\mu^*(E_n) + \frac{\epsilon}{2^n} \right) = \left(\sum_n \mu^*(E_n) \right) + \epsilon.$$

Then we get $\mu^*(\bigcup E_n) \leq \sum \mu^*(E_n)$, and hence μ^* is an outer measure.

Obviously A covered by $A_1 = A$, $A_2, A_3, \ldots = \varnothing$ implies $\mu^*(A) \leq \mu_0(A)$. Now show $\mu_0(A) \leq \mu^*(A)$. If (A_k) covers A, can consider $A = \bigcup (A \cap A_k)$ such that $\mu_0(A) \leq \sum \mu_0(A \cap A_k) \leq \sum \mu_0(A_k)$.

Every $A \in \mathscr{A}$ is μ^*-measurable. For $\mu^*(Q) \geq \mu^*(Q \cap A) + \mu^*(Q \setminus A)$. There exists $A_k \in \mathscr{A}$ with $Q \subset \bigcup_{k=1}^{\infty} A_k$ and $\sum_{k=1}^{\infty} \mu_0(A_k) \leq \mu^*(Q) + \epsilon$ implying $Q \cap A \subset \bigcup_{k=1}^{\infty} (A_k \cap A)$. Also $Q \setminus A \subset \bigcup_{k=1}^{\infty} (A_k \setminus A)$. Q is covered by $A_k \cap A$, $A_k \setminus A$. Hence,

$$\mu^*(Q \cap A) + \mu^*(Q \setminus A) \subseteq \sum \mu_0(A_k \cap A) + \sum_{k=1}^{\infty} \mu_0(A_k \setminus A)$$

where RHS equals

$$\sum_{k=1}^{\infty} (\mu_0(A_k \cap A) + \mu_0(A_k \setminus A)) = \sum \mu_0(A_k) \leq \mu^*(Q) + \epsilon.$$

□

Theorem 7.9. (Translation invariance). *Let $(\mathbb{R}, \mathscr{S}, \mu)$ be Lebesgue measure. If $E \in \mathscr{S}$, $x \in \mathbb{R}$ then $E + x \in \mathscr{S}$ and $\mu(E + x) = \mu(E)$.*

Proof. Show μ^*, the Lebesgue outer measure, is translation invariant. Note $E \subseteq \bigcup_{j=1}^{\infty} I_j$, where I_j are intervals, iff $E + x \subseteq \bigcup_{j=1}^{\infty} (I_j + x)$. Furthermore, get $\sum \mu(I_j)$ and $\sum \mu(I_j + x)$, which are equal.

Now, $E \in \mathscr{S}$ iff E is μ^*-measurable. That is, $\mu^*(Q) \geq \mu^*(Q \cap E) + \mu^*(Q \setminus E)$ iff

$$\mu^*(Q + x) \geq \mu^*(Q \cap E + x) + \mu^*(Q \setminus E + x) = \mu^*((Q + x) \cap (E + x)) + \mu^*((Q + x) \setminus (E + x)).$$

Hence, $E + x$ is μ^*-measurable, i.e., $E + x \in \mathscr{S}$. $\mu(E + x) = \mu^*(E + x) = \mu^*(E) = \mu(E)$. □

Note. Countable union of compact sets is a F_σ-set, and countable intersection of open sets is a G_δ-set.

Theorem 7.10. (Regularity). *If $E \in \mathscr{S}$, then*

(1) $\mu(E) = \inf\{\mu(U) \mid E \subset U, \text{ where } U \text{ is open}\}$;
(2) $\mu(E) = \sup\{\mu(K) \mid K \subseteq E, \text{ where } K \text{ is compact}\}$.

Proof. Let $E \in \mathscr{S}$. Then for all $\epsilon > 0$ there exists I_j such that $E \subseteq \bigcup I_j$ with $\sum_{j=1}^{\infty} \mu(I_j) < \mu(E) + \epsilon$. For I_j with endpoints a_i, b_i, define $\tilde{I_j} = \left(a_j - \frac{\epsilon}{2^j}, b_j + \frac{\epsilon}{2^j}\right)$. So, $E \subseteq U = \bigcup \tilde{I_j}$, where

$$\sum_{j=1}^{\infty} \mu\left(\tilde{I_j}\right) = \sum \left(\mu(I_j) + \frac{2\epsilon}{2^j}\right) = \sum \mu(I_j) + 2\epsilon < \mu(E) + 3\epsilon.$$

So, $\mu(E) \leq \mu(U) < \mu(E) + 3\epsilon$, implying $\mu(U) = \mu(E)$.

Let $E_n = [-n, n] \cap E$. Then $E = \bigcup E_n$. Let $D_n = [-n, n] \setminus E_n$, so $[-n, n] = E_n \cup D_n$. Choose (by (i)) U_n open with $D_n \subseteq U_n$ with $\mu(U_n) < \mu(D_n) + \frac{1}{n}$. Let $K_n = [-n, n] \setminus U_n$, which is compact since it is closed and bounded in $\mathbb{R}$. Hence,

$$\begin{aligned}
\mu(K_n) &= \mu([-n, n]) - \mu([-n, n] \cap U_n) \\
&\geq \mu([-n, n]) - \mu(U_n) \\
&= 2n - \mu(U_n) \\
&\geq 2n - \mu(D_n) - \frac{1}{n} \\
&= 2n - (2n - \mu(E_n)) - \frac{1}{n} = \mu(E_n) - \frac{1}{n}.
\end{aligned}$$

Thus, $\mu(E_n) \leq \mu(K_n) + \frac{1}{n}$. Recall we have $\mu(E) = \lim \mu(E_n)$, since E_n expands to E, so $\mu(E) \leq \limsup \left(\mu(K_n) + \frac{1}{n}\right) \leq \limsup (K_n)$. So, there exists subsequence such that $\mu(K_n) \to \mu(E)$. Hence, $\mu(E)$ is the supremum of $\mu(K_n)$, since $K_n \subseteq E$ yields $\mu(K_n) \leq \mu(E)$. □

7.1. **Uniqueness of Lebesgue measure.** Let G be locally compact topological group. Let $\mathscr{B}$ be the σ-algebra generated by compact sets. μ a measure on $\mathscr{B}$ which is (left) translation invariant if $\mu(aE) = \mu(E)$ for all $a \in G$.

There is a unique non-zero (left Haar) measure with the following properties:

- $\mu(K)$ is finite for K compact
- left translation invariant
- outer regular
- inner regular

Example 7.11. Circle group $\mathbb{R}/2\pi\mathbb{R}$. Finite group: the counting measure is Haar measure. Orthogonal group $O(n)$ of $n \times n$ orthogonal matrices.

Consider two measure spaces $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, \nu)$. The *product measure* of $D \times E \subset E \times Y$ is $\mu(D)\nu(E)$. Let $\mathscr{R} = \{D \times E \mid D \in \mathscr{S}, E \in \mathscr{T}\}$. Let $\mathscr{A}$ be the algebra generated by $\mathscr{R}$ (i.e., all finite unions of rectangles). Check: show that all finite unions of rectangles forms an algebra. Clearly closed under finite unions by definition. Consider a finite union of rectangles $\bigcup_{j=1}^{n} R_j$, $R_j = D_j \times E_j$. Then $(X \times Y) \setminus \bigcup_{j=1}^{n} R_j = \bigcap_{j=1}^{n} (X \times Y \setminus R_j) = \bigcap_{j=1}^{n} ((X \setminus D_j) \times Y \cup X \times (Y \setminus E_j))$.

Define measure π_0 on $\mathscr{A}_0$ by $\pi_0(D \times E) := \mu(D)\nu(E)$. This is a measure on $\mathscr{A}_0$ by the following Lemma.

Lemma 7.12. *Let $\{D_j \times E_j\}$ be a sequence of disjoint rectangles in $X \times Y$. If $\bigcup_{j=1}^{\infty} D_j \times E_j = \bigcup_{k=1}^{n} F_k \times G_k$, where $F_k \times G_k$ rectangles in $X \times Y$, then $\sum_{j=1}^{\infty} \pi_0(D_j \times E_j) = \sum_{k=1}^{n} \pi_0(F_k \times G_k)$.*

Proof. We know $\sum_{j=1}^{\infty}\chi_{D_j\times E_j}(x,y)=\sum_{k=1}^{n}\chi_{F_k\times G_k}(x,y)$. Therefore,

$$\sum_{j=1}^{\infty}\chi_{D_j}(x)\chi_{E_j}(y)=\sum_{k=1}^{n}\chi_{F_k}(x)\chi_{G_k}(y).$$

Fix y; each $\chi_D(x)$ is $\mathscr{S}$-measurable. Integrate w.r.t. μ (Beppo Levi)

$$\int_X \chi_D(x)d\mu(x)=\mu(D)\Longrightarrow\sum_{j=1}^{\infty}\mu(D_j)\chi_{E_j}(y)=\sum_{k=1}^{n}\mu(F_k)\chi_{G_k}(y).$$

Again, integrate w.r.t. ν (Beppo Levi), so

$$\sum_{j=1}^{\infty}\pi_0(D_j\times E_j)=\sum_{j=1}^{\infty}\mu(D_j)\nu(E_j)=\sum_{k=1}^{n}\mu(F_k)\nu(G_k)=\sum_{k=1}^{n}\pi_0(F_k\times G_k).$$

□

π_0 is the measure on the algebra of rectangles, so

$$\pi_0^*(E)=\inf\left\{\sum_{j=1}^{\infty}\pi_0(D_j\times E_j)\;\middle|\;E\subseteq\bigcup_{j=1}^{\infty}D_j\times E_j, D_j\in\mathscr{S},E_j\in\mathscr{T}\right\}.$$

So, we get a complete measure by Carathedory (exists a complete measure on a σ-algebra $\mathscr{U}$ with $\mathscr{A}_0\subseteq\mathscr{U}$ and the measure on sets in the algebra is given by π_0). Note "$\mathscr{U}$ is difficult to describe". The σ-algebra generated by the rectangles (or $\mathscr{A}_0$) is contained in $\mathscr{U}$, and is defined to be the *product σ-algebra.* $\mathscr{S}\star\mathscr{T}:=\sigma$-algebra generated by the rectangles. The product measure is defined to be $\pi=\mu\star\nu$, the restriction of π^* to $\mathscr{S}\star\mathscr{T}$.

We may consider $Q\subset X\times Y$ and wish to integrate $\int_{X\times Y}f(x,y)d\pi(x,y)$ (where $\pi=\mu\star\nu$). We ask, is this integral equal to $\int_X\int_Y f(x,y)d\nu(y)d\mu(x)$.

Define the *x-section* to be $Q_x:=\{y\in Y\mid(x,y)\in Q\}$ and the *y-section* to be $Q^y:=\{x\in X\mid(x,y)\in Q\}$.

Lemma 7.13. (1) *If $Q\in\mathscr{S}\star\mathscr{T}$, then $Q_x\in\mathscr{T}$ and $Q^y\in\mathscr{S}$.*
(2) *If $f:X\times Y\to\overline{\mathbb{R}}$ is $\mathscr{S}\star\mathscr{T}$-measurable, then $f(\cdot,y)$ is $\mathscr{S}$-measurable and $f(x,\cdot)$ is $\mathscr{T}$-measurable.*

Proof. Let $\mathscr{V}=\{V\subseteq X\times Y\mid V_x\in\mathscr{T},\forall x\in X\}$. Want $\mathscr{S}\star\mathscr{T}\subseteq\mathscr{V}$. Show $\mathscr{V}$ is σ-algebra that contains the rectangles and hence $\mathscr{S}\star\mathscr{T}$.

$\mathscr{V}$ is a σ-algebra: $(\varnothing\times\varnothing)_x=\varnothing$ and $(X\times Y)_x=Y$ both in $\mathscr{T}$. Suppose $V\in\mathscr{V}$. Then $((X\times Y)\setminus V)_x=Y\setminus V_x$, where $V_x\in\mathscr{T}$ so certainly $Y\setminus V_x\in\mathscr{T}$. Hence, $X\times Y\setminus V\in\mathscr{V}$. Suppose (V_n) sequence of $\mathscr{V}$. Then $(\bigcup V_n)_x=\bigcup(V_n)_x\in\mathscr{T}$, so $\bigcup V_n\in\mathscr{V}$. Hence, $\mathscr{V}$ is a σ-algebra.

Now, $(D\times E)_x=E\in\mathscr{T}$, proving (1) (where $D\times E$ is a rectangle).

Suppose $f:X\times Y\to\overline{\mathbb{R}}$ is $\mathscr{S}\star\mathscr{T}$-measurable. Fix y, $E_y=\{x\mid f(x,y)>\alpha\}$ where $\alpha\in\mathbb{R}$. Observe $E_y=\{(x,y)\mid f(x,y)>\alpha\}^y=Q^y$ where $Q\in\mathscr{S}\star\mathscr{T}$ and this section is in $\mathscr{S}$ by (1), proving (2). □

Example 7.14. Let m_d be Lebesgue measure on $\mathbb{R}^d$ and $\mathscr{S}_d$ be the Lebesgue σ-algebra (this is complete). Let $E\subset[0,1]$ be a non-Lebesgue measurable subset of $\mathbb{R}$ (e.g., a Vitali set). Then in $\mathbb{R}^2$, consider $E\times\{0\}$ which is $\mathscr{S}_2$-measurable (subset of $[0,1]\times\{0\}$ which has zero Lebesgue measure). But $(E\times\{0\})^y=E$ is not measurable. So, $E\times\{0\}$ is not in $\mathscr{S}_1\star\mathscr{S}_1$ implying $\mathscr{S}_1\star\mathscr{S}\subsetneq\mathscr{S}_2$.

Theorem 7.15. (Tonelli) *Let $(X,\mathscr{S},\mu)$ and $(Y,\mathscr{T},\nu)$ be σ-finite measurable spaces with $(X\times Y,\mathscr{S}\star\mathscr{T},\mu\star\nu)$ and $f:X\times Y\to\overline{\mathbb{R}}$ non-negative and $\mathscr{S}\star\mathscr{T}$-measurable. Then*

(1) *The function*

$$y \mapsto \int_X f(x,y)d\mu(x)$$

is $\mathscr{T}$-measurable, and

$$\int_Y \int_X f(x,y)d\mu(x)d\nu(y) = \int_{X\times Y} f(x,y)d\pi(x,y)$$

where $d\pi(x,y) = d\mu \star \nu(x,y)$.

(2) *The function*

$$x \mapsto \int_Y f(x,y)d\nu(y)$$

is $\mathscr{S}$-measurable and

$$\int_X \int_Y f(x,y)d\nu(y)d\mu(x) = \int_{X\times Y} f(x,y)d\pi(x,y).$$

Hence, the related integrals are equal.

Proof. Let $f = \chi_Q$, Q rectangle. $\mu(X), \nu(Y) < \infty$.

We now reduce to a simpler case. Suppose theorem holds for (f_n) increasing, then it holds for $f = \lim f_n$. By monotone convergence theorem, $x \mapsto \int_Y f_n(x,y)d\mu$ is increasing, so $\lim \int_X \int_Y f_n d\nu d\mu = \int_X \int_Y f d\nu d\mu$. Since $f_n \to f$, by MCT

$$\lim \int_{X\times Y} f_n d(\mu\star\nu) = \int_{X\times Y} f d(\mu\star\nu) = \lim \int_X \int_Y f_n d\nu d\mu = \int_X \int_Y f d\nu d\mu.$$

Claim: if theorem holds for μ and ν finite, then it holds μ and ν σ-finite. Let μ and ν be σ-finite, i.e., $X = \bigcup_n X_n$ $\mu(X_n) < \infty$ expanding union and $Y = \bigcup_n Y_n$ $\nu(Y_n) < \infty$ expanding. Let $f_n = f\chi_{X_n\times Y_n}$. Then (f_n) is non-negative, measurable and increasing with limit f.

The integrals for f_n are integrals w.r.t. $\mu\mid_{X_n}$ and $\nu\mid_{Y_n}$ (both finite) and $\mu\star\nu\mid_{X_n\times Y_n}$

$$\int_X f_n d\mu(x) = \int_X f\chi_{X_n\times Y_n} d\mu = \chi_{Y_n}\int_{X_n} f d\mu,$$

proving our claim. Hence, we need only prove Tonelli for finite measures.

Claim: We need only prove Tonelli for characteristic functions. There exists an increasing sequence of non-negative measurable functions $f_n \to f \geq 0$. $f_n = \sum_{j=1}^{k_n} a_j^{(n)}\chi_{A_j(n)}$. If Tonelli holds for characteristic functions (measurable), then it holds for simple functions, and hence all functions.

Without loss of generality, suppose μ, ν are finite and $f = \chi_Q$, $Q \in \mathscr{S}\star\mathscr{T}$. Consider $\mathscr{Q} = \{Q \in \mathscr{S}\star\mathscr{T} \mid \text{Tonelli holds for } \chi_Q\}$ (aim to show $\mathscr{Q} = \mathscr{S}\star\mathscr{T}$). Need to show $\mathscr{Q}$ is a σ-algebra and $\mathscr{Q}$ contains all rectangles. Suppose $D \times E = Q$ is a rectangle, and let $f = \chi_Q$. Then

$$\int_Y f(x,y)d\nu(y) = \int_Y \chi_D(x)\chi_E(y)d\mu(y) = \chi_D(x)\nu(E).$$

So,

$$\begin{aligned}\int_X \int_Y f(x,y)d\nu(y)d\mu(x) &= \int_X \chi_D(x)\nu(E)d\mu \\ &= \nu(E)\int_X \chi_D(x)d\mu = \nu(E)\mu(D) \\ &= (\mu\star\nu)(Q) = \int_{X\times Y}\chi_Q d(\mu\star\nu).\end{aligned}$$

Remains to show $\mathscr{Q}$ is a σ-algebra. To do so, we provide the following definition. □

Definition 7.16. A collection $\mathscr{M}$ of subsets of X is a *monotone class* if it is closed under countable unions and countable intersections.

Example 7.17. All intervals of form $(-a, a)$ in $\mathbb{R}$, this is a monotone class. Remark: All σ-algebras are monotone classes, but not vice versa as example demonstrates.

Lemma 7.18. (Monotone Class Lemma). *A monotone class is a σ-algebra if it contains an algebra which generates the σ-algebra.*

Now, using this, we may finish the proof of Tonelli: it suffices to show $\mathscr{Q}$ is a monotone class, and hence equal $\mathscr{S} \star \mathscr{T}$. Let Q_n be an expanding sequence in $\mathscr{Q}$, i.e., Tonelli holds for χ_{Q_n}. Since $\chi_{\bigcup Q_n} = \lim \chi_{Q_n}$, Tonelli holds for $\chi_{\bigcup Q_n}$ implying $\bigcup Q_n \in \mathscr{Q}$. Let C_n be a contracting sequence of sets in $\mathscr{Q}$. Consider expanding sequence $(X \times Y \setminus C_n)$ with $\bigcup (X \times Y \setminus C_n) = X \times Y \setminus (\bigcap C_n)$. Tonelli holds for this, i.e., $\int_X \int_Y \chi_{X\times Y \setminus \bigcap C_n} d\nu d\mu = \int_{X\times Y} \chi_{X \times Y \setminus \bigcap C_n} d(\mu \star \nu)$. Observe

$$\begin{aligned}
&\chi_{X\times Y\setminus \bigcap C_n} = 1 - \chi_{\bigcap C_n} \\
\implies &\int_X \int_Y (1 - \chi_{\bigcap C_n}) d\nu d\mu = \int_{X\times Y} (1 - \chi_{\bigcap C_n}) d(\mu \star \nu) \\
\implies &\int_X \int_Y 1 d\nu d\mu - \int_X \int_Y \chi_{\bigcap C_n} = \int_{X\times Y} 1 d(\mu\star\nu) - \int_{X\times Y} \chi_{\bigcap C_n} d(\mu \star \nu) \\
\implies &\int_X \int_Y \chi_{\bigcap C_n} = \int_{X\times Y} \int_{\bigcap C_n},
\end{aligned}$$

and thus $\bigcap C_n \in \mathscr{Q}$ and we have finished proof of Tonelli.

Example 7.19. σ-finite condition is necessary. Take $X = \bigcup_n X_n$ where X uncountable. Let μ be Lebesgue measure on $\mathbb{R}$ and ν be counting measure on $\mathbb{R}$. Prior is σ-finite, latter is not. Consider diagonal $D = \{(x, x) \mid x \in \mathbb{R}\}$ and $f = \chi_D$; consider $D_n = \bigcup_x B\big(x, \frac{1}{n}\big) \in \mathscr{S} \star \mathscr{T}$ and intersection over D_n yields D. Now,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) d\mu(x) d\nu(y) = \int_{\mathbb{R}} 0 d\nu = 0$$

and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) d\nu d\mu = \int_{\mathbb{R}} 1 d\mu = \infty.$$

What is $\int_{\mathbb{R}\times\mathbb{R}} f d(\mu \star \nu) = (\mu \star \nu)(D)$.

Theorem 7.20. *Fubini First Theorem. If $f \geq 0$ and μ, ν are σ-finite, then f is product integrable iff one of the integrals of Tonelli (e.g., $\int_X \int_Y |f| d\nu d\mu$) finite.*

We want for Fubini: $\int_X \int_Y f d\nu d\mu = \int_{X\times Y} f d(\mu \star \nu)$. $x \mapsto \int_Y |f| d\nu$ is μ-integrable, i.e., finite μ-a.e.

Theorem 7.21. (Fubini Second Theorem). *μ, ν σ-finite, and $f : X \times Y \to \overline{\mathbb{R}}$ $\mu \star \nu$-measurable.*

(1) *There is a set Y_0, $\nu(Y_0) = 0$ such that $f(\cdot, y)$ is μ-integrable over X for all $y \in Y \setminus Y_0$.*

$$y \mapsto \begin{cases} \int_X f(x, y) d\mu(x) & y \in Y \setminus Y_0; \\ 0 & y \in Y_0; \end{cases}$$

is ν-integrable with

$$\int_Y \int_X f(x, y) d\mu d\nu = \int_{X\times Y} f(x, y) d(\mu \star \nu).$$

Example 7.22. If $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}|a_{nm}|<\infty$, then $\sum_n\sum_m a_{nm}=\sum_m\sum_n a_{nm}\in\mathbb{R}$ (of Fubini).

Fubini theorem: Suppose $f:X\times Y\to\overline{\mathbb{R}}$ is $\mu\star\nu$-integrable. Then $\int_{X\times Y}|f|d(\mu\star\nu)<\infty$ (where integral equal to $\int_X\int_Y|f|d\nu d\mu=\int_Y\int_X|f|d\mu d\nu$ by Tonelli).

Example continued: $\mu=\nu$ are counting measure on $\mathbb{N}$. $f:\mathbb{N}\times\mathbb{N}\to\overline{\mathbb{R}}$ $(n,m)\mapsto a_{nm}$. We get $\int_{\mathbb{N}}\int_{\mathbb{N}}f(n,m)d\mu(n)d\nu(m)$ with $\int_{\mathbb{N}\times\mathbb{N}}fd(\mu\star\nu)=\sum_n\sum_m a_{nm}=\sum_m\sum_n a_{nm}\in\mathbb{R}$.

8. Extra Topics

8.1. **Radon-Nikodym.** Let μ be a measure and $f\geq 0$ measurable. Then λ given by $\lambda(E):=\int_E fd\mu$, $E\in\mathscr{S}$ is a measure on $\mathscr{S}$. This λ has the property: $\mu(E)=0$ implies $\lambda(E)=0$. We say λ is absolutely continuous with respect to μ, written $\lambda<<\mu$.

Lemma 8.1. *If λ,μ are finite measures, then $\lambda<<\mu$ if and only if for each $\epsilon>0$ there exists $\delta>0$ such that $\lambda(E)<\epsilon$ whenever $\mu(E)<\delta$.*

Theorem 8.2. (Radon-Nikodym Theorem). *If μ and λ are σ-finite with $\lambda<<\mu$, then there exists $f\geq 0$ measurable, with $\lambda(E)=\int_E fd\mu$.*

Note. f is called the *Radon-Nikodym derivative* of λ w.r.t. μ, sometimes written $f=\frac{d\lambda}{d\mu}$.

Definition 8.3. Let $(X,\mathscr{S})$ be a measurable space. Then $\lambda:\mathscr{S}\to\mathbb{R}$ is a *charge* (or signed measure) if $\lambda(\varnothing)=0$ and λ is countably additive (i.e., $\lambda(\bigcup E_n)=\sum\lambda(E_n)$).

Example 8.4. If $f\in L_2(\mu)$, then $\lambda(E)=\int_E fd\mu$ is a charge. We say $P\in\mathscr{S}$ is *positive* for λ if $\lambda(P\cap E)\geq 0$, $E\in\mathscr{S}$. Similarly *negative* and *null*.

8.2. **Hahn-decomposition.** If λ is a charge, then there exists P positive N negative with disjoint union X, which is unique up to a null set. Choose $A_n\to\sup\{\lambda(A)\mid A\in\mathscr{S},A\text{ is positive}\}$. $P=\bigcup_{n=1}^{\infty}A_n$.

Definition 8.5. Let λ be a charge, P, N be the Hahn-decomposition up to positive and negative parts. Then $\lambda^+(E)=\lambda(E\cap P)$, $\lambda^-(E)=\lambda(E\cap N)$ are finite measures with $\lambda=\lambda^+-\lambda^-$ (Hahn-decomposition). Call λ^+ *positive part* of λ and λ^- *negative part* of λ. Then the *total variation* of λ is $|\lambda|=\lambda^++\lambda^-$.

Theorem 8.6. *If $f\in L_1(\mu)$ and $\lambda(E)=\int_E fd\mu$ then $\lambda^+(E)=\int_E f^+d\mu$, $\lambda^-(E)=\int_E f^-d\mu$, $|\lambda|(E)=\int_E|f|d\mu$.*

$\lambda:\mathscr{S}\to\mathbb{C}$ has $(\Re(\lambda))(E):=\Re(\lambda(E))$ and $(\Im(\lambda))(E):=\Im(\lambda(E))$. Take $\lambda=(\Re(\lambda))+i(\Im(\lambda))$, where $\Re(\lambda)$ and $\Im(\lambda)$ are charges. Also, $f:X\to\mathbb{C}$ measurable iff $\Re(f)$, $\Im(f)$ measurable.

Define a *positive variation* of such a $|\lambda|(A)=\sup\sum_{n=1}^{\infty}\lambda(A_n)$. This defines a positive measure $\left|\int_X fd\lambda\right|\leq\int|f|d|\lambda|$. The Radom-Nikodym theorem holds for real and complex measures.

$\frac{d(\nu+\lambda)}{d\mu}=\frac{d\nu}{d\mu}+d\lambda/d\mu$, $\nu,\lambda<<\mu$: μ-a.e., $\lambda<<\nu<<\mu$, $\frac{d\lambda}{d\mu}=\frac{d\lambda}{d\nu}\frac{d\nu}{d\mu}$. $\lambda<<\mu$ and $\mu<<\lambda$, $\frac{d\mu}{d\lambda}=\left(\frac{d\lambda}{d\mu}\right)^{-1}$ λ-a.e. Can define vector consisting of such measures, over Banach (and indeed locally convex spaces).

We say λ, μ are *mutually singular* if exists disjoint sets A and B union X for which $\lambda(A)=\mu(B)=0$, which we write as $\lambda\perp\mu$.

Theorem 8.7. Lebesgue decomposition. *Let λ, μ be σ-finite measures. Then λ can be written uniquely $\lambda=\lambda_1+\lambda_2$ where $\lambda_1\perp\mu$, $\lambda_2<<\mu$.*

Proof. Let $\nu = \lambda + \mu$ implying $\lambda, \mu << \nu$. By Radon-Nikodyn $\lambda(E) := \int_E f d\nu$ and $\mu(E) = \int_E g d\nu$. Let $A = \{x \mid g(x) = 0\}$ (λ concentrated) and $B = \{x \mid g(x) > 0\}$ (μ concentrated). $X = A \cup B$ (disjoint union), unique up to a set of ν-measure zero. Let $\lambda_1(E) = \lambda(E \cap A)$ and $\lambda_2(E) = \lambda(E \cap B) = \lambda_2(E \cap (X \setminus A))$. Clearly λ_1, λ_2 are measures with $\lambda = \lambda_1 + \lambda_2$.

Need $\lambda_1 \perp \mu$. Observe

$$\mu(A) = \int_A g d\nu = \int 0 d\nu = 0$$

and

$$\lambda_1(B) = \lambda(B \cap A) = \lambda(\varnothing) = 0.$$

Now, need $\lambda_2 << \mu$. Suppose $\mu(E) = 0$, so $\mu(E) = \int_E g d\nu = 0$, so $g = 0$ ν-a.e. on E. $\lambda_2(E) = \lambda(E \cap B)$, where $E \subseteq A$ since $\int_{E \cap B} g d\nu = 0$, and claim follows. □